Problem 11217. Proposed by Michel Bataille, Rouan, France. For a positive integer let $S_n$ denote the set of all numbers of the form
\[
x^n y^m z^n \frac{y^n}{y^n-1} \frac{z^n}{z^n-1} \frac{x^n}{x^n-1}
\]such that $x$, $y$ and $z$ are positive numbers, each different from 1, with $xyz = 1$. Show that $S_n$ is bounded below, and find the greatest lower bound of $S_n$ in terms of $n$.

Solution, by Omran Kouba (Higher Institute for Applied Sciences And Technology, Damascus, Syria).

The answer is $\inf S_n = 1 + 9 \left( \frac{n}{2} \right)$. Let us define $f_n(x, y, z)$ by
\[
f_n(x, y, z) = \frac{x^n}{y^n-1} \frac{y^n}{z^n-1} \frac{z^n}{x^n-1}
\]where $n$ is a positive integer, $x$, $y$ and $z$ are positive numbers, each different from 1. It is a trivial to see that
\[
f_1(x, y, z) = \frac{x}{y-1} + \frac{y}{z-1} + \frac{z}{x-1}
\]so, if $xyz = 1$ then $f_1(x, y, z) = 1$, that is $S_1 = \{1\}$. In order to study the general case, we will use the following lemma where we express $f_n(x, y, z)$ differently.

Lemma. If $xyz = 1$ then
\[
f_{n+1}(x, y, z) = A_{n-2}(x, y, z) + B(x, y, z) A_{n-1}(x, y, z) + A_n(x, y, z)
\]where
\[
A_n(x, y, z) = \sum_{k+\ell+m=n} \left( \frac{x}{y} \right)^k \left( \frac{y}{z} \right)^\ell \left( \frac{z}{x} \right)^m
\](in particular $A_{-1}(x, y, z) = 0$, $A_0(x, y, z) = 1$) and
\[
B(x, y, z) = 1 + x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.
\]

- Now, since $xyz = 1$ and using the fact that $a + \frac{1}{a} \geq 2$ for a positive number $a$, we see that
\[
B(x, y, z) = 1 + x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} \geq 7
\]with equality if $x = y = z = 1$.

- It is clear that the product of the terms in the sum defining $A_n(x, y, z)$ equals 1, so using the arithmetic-geometric mean inequality we find that
\[
A_n(x, y, z) \geq \text{card} \{ (k, \ell, m) \in \mathbb{N}^3 : k + \ell + m = n \} = \binom{n+2}{2}
\]with equality if $x = y = z = 1$.

Then, under the assumption $xyz = 1$ we have
\[
f_{n+1}(x, y, z) \geq \binom{n}{2} + 7 \binom{n+1}{2} + \binom{n+2}{2} = 1 + 9 \binom{n+1}{2}
\]with equality if $x = y = z = 1$. Note that, by the lemma, 1 is not a real singularity of $f_n(x, y, z)$ if $xyz = 1$. 

Proof of the Lemma. We will use generating functions, clearly we have

\[
\sum_{n=0}^{\infty} f_{n+1}(x, y, z)U^n = \frac{x}{1-(x/y)U} + \frac{y}{1-(y/z)U} + \frac{z}{1-(z/x)U} = \alpha + \beta U + \gamma U^2 \frac{(1-(x/y)U)(1-(y/z)U)(1-(z/x)U)}{(1-(x/y)U)(1-(y/z)U)(1-(z/x)U)}
\]

Where \( \alpha, \beta \) and \( \gamma \) are functions of \( x, y \) and \( z \) to be calculated. Using \( xyz = 1 \) we find

\[
\gamma = \frac{y}{(y-1)(1-x)} + \frac{z}{(z-1)(1-y)} + \frac{x}{(x-1)(1-z)} = f_1 \left( \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right) = 1
\]

and finally, noting that under the assumption \( xyz = 1 \), (and may be, by using a computer algebra system,) we have

\[
f_2(x, y, z) = 1 + x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 1
\]

consequently, comparing the coefficient of \( U \), we get

\[
\beta = 1 + x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = B(x, y, z)
\]

So, we have proved

\[
\sum_{n=0}^{\infty} f_{n+1}(x, y, z)U^n = \frac{1 + B(x, y, z)U + U^2}{(1 - \frac{x}{y}U)(1 - \frac{y}{z}U)(1 - \frac{z}{x}U)}.
\]

On the other hand we have

\[
\frac{1}{(1 - \frac{x}{y}U)(1 - \frac{y}{z}U)(1 - \frac{z}{x}U)} = \sum_{n=0}^{\infty} A_n(x, y, z)U^n
\]

and we get the result by comparing the coefficient of \( U^n \) on both sides of the equality

\[
\sum_{n=0}^{\infty} f_{n+1}(x, y, z)U^n = (1 + B(x, y, z)U + U^2) \sum_{n=0}^{\infty} A_n(x, y, z)U^n.
\]

This ends the proof of the lemma.
Problem 11219. Proposed by R. A. STRUBEZ, Santa Monica, CA. Prove that when \( n \) is a positive integer and \( s \) is a real number greater than 1

\[
1 + n(\zeta(s) - 1) \leq \sum_{k=0}^{\infty} \left( \frac{n}{n+k} \right)^s \leq n\zeta(s).
\]

Solution, by OMAR KOUBA (Higher Institute for Applied Sciences And Technology, Damascus, Syria).

Using the Euler gamma function defined by \( \Gamma(s) = \int_0^\infty u^{s-1}e^{-u}du \) for \( s > 0 \) we find, by simple change of variable \( u \leftarrow (1 + \frac{k}{n})t \), that

\[
\left( \frac{n}{n+k} \right)^s = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}e^{-t} \exp \left( -\frac{kt}{n} \right) dt.
\]

Noting that \( (k, t) \mapsto t^{s-1}e^{-t} \exp \left( -\frac{kt}{n} \right) \) is positive, we can sum over \( k \) to find that, for \( s > 1 \) and \( n \in \mathbb{N}^* \), we have

\[
\sum_{k=0}^{\infty} \left( \frac{n}{n+k} \right)^s = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{e^{-t}}{1 - e^{-t/n}} \frac{1 - e^{-t}}{1 - e^{-t/n}} dt
\]

and

\[
\sum_{k=1}^{\infty} \left( \frac{n}{n+k} \right)^s = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{e^{-t}}{e^t - 1} \frac{e^t - 1}{e^{t/n} - 1} dt.
\]

In particular, for \( n = 1 \) we get

\[
\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{e^{-t}}{1 - e^{-t}} dt \quad \text{and} \quad \zeta(s) - 1 = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{e^{-t}}{e^t - 1} dt
\]

So the desired inequalities are direct consequences of the following two inequalities

\[
\frac{1 - e^{-t}}{1 - e^{-t/n}} = \sum_{j=0}^{n-1} e^{-jt/n} \leq n \quad \text{and} \quad \frac{e^t - 1}{e^{t/n} - 1} = \sum_{j=0}^{n-1} e^{jt/n} \geq n.
\]

which are clearly true if \( t > 0 \) and \( n \in \mathbb{N}^* \).
Problem 11220. Proposed by David Beckwith, Sag Harbor, NY. Show that when \( n \) is a positive integer

\[
\sum_{r=0}^{n} (-1)^r \binom{n}{r} \left( \frac{2n - 2r}{n - 1} \right) = 0.
\]

Solution, by Omran Kouba (Higher Institute for Applied Sciences And Technology, Damascus, Syria).

This is a simple consequence of the following result.

**Lemma.** If \( P \) is a complex polynomial such that \( \deg P < n \) then

\[
\sum_{r=0}^{n} (-1)^r \binom{n}{r} P(r) = 0.
\]

because the polynomial \( Q(X) = \frac{1}{(n-1)!} (2X)(2X-1)\cdots(2X-n+2) \) is of degree \( n - 1 \), and the desired sum can be written in the following way

\[
\sum_{r=0}^{n} (-1)^r \binom{n}{r} \left( \frac{2n - 2r}{n - 1} \right) = (-1)^n \sum_{r=0}^{n} (-1)^r \binom{n}{r} \left( \frac{2r}{n - 1} \right) = (-1)^n \sum_{r=0}^{n} (-1)^r \binom{n}{r} Q(r).
\]

**proof of the lemma.** It is enough to prove it when \( P = X^k \) for \( 0 \leq k < n \), and to do this, we note that

\[
\sum_{r=0}^{n} (-1)^r \binom{n}{r} e^{rz} = (1 - e^z)^n
\]

so \( z_0 = 0 \) is a zero of order \( n \) of the analytic fonction \( z \mapsto f(z) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} e^{rz} \) and consequently \( f^{(k)}(0) = 0 \) for \( 0 \leq k < n \) and this means that

\[
\forall k \in \{0, 1, \ldots, n - 1\}, \quad \sum_{r=0}^{n} (-1)^r \binom{n}{r} r^k = 0
\]

and this finishes the proof of the lemma.

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Problem 11248. Proposed by Pál Péter Dályay, Deák Fernek Hight School, Szeged, Hungary. Let \( n \) be a positive integer, and let \( f \) be continuous real-valued function on \([0, 1]\) with the property that \( \int_0^1 x^k f(x) \, dx = 1 \) for \( 0 \leq k \leq n - 1 \). Prove that \( \int_0^1 (f(x))^2 \, dx \geq n^2 \).

Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

We will use Legendre polynomials, defined, for \( n \in \mathbb{N} \), by \( \ell_n(X) = \frac{1}{n!} \frac{d^n}{dX^n} ((X^n(X - 1)^n) \). In fact, we will make use of the following two properties:

1. Orthogonality: for \((n, m) \in \mathbb{N}^2\), we have \( \int_0^1 \ell_n(x)\ell_m(x) \, dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{1}{2n+1} & \text{if } n = m \end{cases} \)

2. for \( n \in \mathbb{N} \), we have \( \ell_n(1) = 1 \).

We consider the space \( E = C([0, 1]) \) of real-valued continuous functions on \([0, 1]\) equipped with the inner product defined by \( \langle f, g \rangle = \int_0^1 fg \), and the subspace \( F_n \) of polynomial functions of degree \( \leq n - 1 \), recall that \((\ell_0, \ell_1, \ldots, \ell_{n-1})\) in an orthogonal basis in \( F_n \).

Consider a function \( f \) satisfying \( \int_0^1 x^k f(x) \, dx = 1 \) for \( 0 \leq k \leq n - 1 \). using linearity, this is equivalent to

\[
\forall P \in F_n, \quad \langle f, P \rangle = P(1)
\]

In particular, using property 2., we have

\[
\forall k \in \{0, 1, \ldots, n - 1\}, \quad \langle f, \ell_k \rangle = \ell_k(1) = 1
\]

This proves that the orthogonal projection of \( f \) on the subspace \( F_n \) is

\[
P_n(f) = \sum_{k=0}^{n-1} (2k + 1) \langle f, \ell_k \rangle \ell_k = \sum_{k=0}^{n-1} (2k + 1) \ell_k
\]

from this we conclude that

\[
\|P_n(f)\|^2 = \sum_{k=0}^{n-1} (2k + 1)^2 \|\ell_k\|^2 = \sum_{k=0}^{n-1} (2k + 1) = n^2.
\]

Now, since \( \|P_n(f)\|^2 + \|f - P_n(f)\|^2 = \|f\|^2 \) we conclude that \( \|f\|^2 \geq n^2 \) with equality if and only if \( f = \sum_{k=0}^{n-1} (2k + 1) \ell_k \).

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Problem 11259. Proposed by Nobuhisa Abe, NBU Attached Senior Hight School, Saiki, Japan. For integers $n$ greater than 2, let

$$f(n) = \sum_{j=1}^{n-2} 2^j \sum_{S \subseteq \{1, \ldots, n-1\}, \text{card}(S) = j} \prod_{k \in S} k$$

where the sum is over all $j$-element subsets $S$ of the set $\{1, 2, \ldots, n-1\}$. Show that $4(2n-1)! + (f(n))^2$ is never the square of an integer.

Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

We will use the following fact which is well-known and can be proved by induction,

$$(1 + a_k)^m = \sum_{S \subseteq \mathbb{N}_m} \prod_{k \in S} a_k$$

where $\mathbb{N}_m = \{1, 2, \ldots, m\}$. (with the convention $\prod_{k \in \emptyset} a_k = 1$.)

So we can rewrite $f(n)$ in the following way

$$f(n) = \sum_{j=1}^{n-2} 2^j \sum_{S \subseteq \mathbb{N}_{n-1}, \text{card}(S) = j} \prod_{k \in S} k$$

$$= \sum_{j=1}^{n-2} \sum_{S \subseteq \mathbb{N}_{n-1}, \text{card}(S) = j} \prod_{k \in S} (2k)$$

$$= \sum_{\emptyset \subseteq S \subseteq \mathbb{N}_{n-1}, \text{card}(S) = j < n-1} \prod_{k \in S} (2k)$$

$$= \left( \sum_{S \subseteq \mathbb{N}_{n-1}} \prod_{k \in S} (2k) \right) - \prod_{k \in \mathbb{N}_{n-1}} (2k) - \prod_{k \in \emptyset} (2k)$$

$$= \left( \prod_{k \in \mathbb{N}_{n-1}} (1 + 2k) \right) - \left( \prod_{k \in \mathbb{N}_{n-1}} (2k) \right) - 1$$

We conclude that $f(n) = A_n - B_n - 1$, with $A_n = 1 \cdot 3 \cdot 5 \cdots (2n-1)$ and $B_n = 2 \cdot 4 \cdot 6 \cdots (2n-2)$. So if we define $g(n) = 4(2n-1)! + (f(n))^2$ we find that

$$g(n) = 4A_nB_n + (A_n - B_n - 1)^2$$

$$= (A_n + B_n)^2 - 2(A_n - B_n) + 1$$

From this, we see directly that

$$(A_n + B_n)^2 - g(n) = 2(A_n - B_n - 1) + 1 = 2f(n) + 1 > 0$$

$$g(n) - (A_n + B_n - 1)^2 = 4B_n > 0$$

and this proves that $(N-1)^2 < g(n) < N^2$ with $N = A_n + B_n$. So $g(n)$ can not be the square of an integer.

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Problem 11260. Proposed by PAOLO PERFETTI, Mathematics Department, University “Tor Vergata,” Rome, Italy. Find those nonnegative values of $\alpha$ and $\beta$ for which

$$\sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{\alpha + k \log k}{\beta + (k + 1) \log(k + 1)}$$

converges. For those values of $\alpha$ and $\beta$, evaluate the sum.

Solution, by OMRRAN KOUBA (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

The answer is: the series converges iff ($\alpha = 0$) or ($\beta > \alpha > 0$) and then the sum is 0 if $\alpha = 0$ and $\alpha/(\beta - \alpha)$ otherwise.

- The case $\alpha = 0$ is easy since, then, all the terms of series are zero.
- Next we assume that $\alpha > 0$. Let’s define the general term sequence $(u_n)_{n \geq 0}$ by

$$u_0 = 1 \text{ and } \forall n \geq 1, \quad u_n = \prod_{k=1}^{n} \frac{\alpha + k \log k}{\beta + (k + 1) \log(k + 1)}$$

All these terms are positive, and we have

$$\forall n \geq 1, \quad \frac{u_n}{u_{n-1}} = \frac{\alpha + n \log n}{\beta + (n + 1) \log(n + 1)}$$

and this can be rearranged in the following way

$$\forall n \geq 1, \quad (\alpha - \beta)u_n = (\alpha + (n + 1) \log(n + 1))u_n - (\alpha + n \log n)u_{n-1}$$

Summing these equalities as $n$ varies from 1 to $m$ we find that

$$\forall m \geq 1, \quad (\alpha - \beta) \sum_{n=1}^{m} u_n = (\alpha + (m + 1) \log(m + 1))u_m - \alpha$$

(*)

- If $\alpha \geq \beta$ then from (*) we conclude that $(\alpha + (m + 1) \log(m + 1))u_m - \alpha \geq 0$ for every $m$ that is

$$\forall m \geq 1, \quad u_m \geq \frac{\alpha}{\alpha + (m + 1) \log(m + 1)}$$

and the series $\sum u_n$ diverges since the well-known Bertrand series $\sum \frac{1}{n \log n}$ does.

- If $\alpha < \beta$ then from (*) we conclude that

$$\forall m \geq 1, \quad \sum_{n=1}^{m} u_n = \frac{\alpha - (\alpha + (m + 1) \log(m + 1))u_m}{\beta - \alpha} < \frac{\alpha}{\beta - \alpha}$$

and this proves the convergence of the series $\sum_{n=1}^{\infty} u_n$. Let $\ell$ be the sum of this series. Using (*) again we see that in this case we have

$$\lim_{m \to \infty} (\alpha + (m + 1) \log(m + 1))u_m = \alpha + (\alpha - \beta)\ell$$

If $\alpha + (\alpha - \beta)\ell \neq 0$ then the two series $\sum u_n$ and $\sum \frac{1}{n \log n}$ would have the same nature, and this contradicts the convergence of $\sum u_n$. So we must have $\alpha + (\beta - \alpha)\ell = 0$ that is $\ell = \alpha/(\beta - \alpha)$. 

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Problem 11262. Proposed by Ashay Burungale, Satara, Maharashtra, India. In a certain town of population $2n + 1$, one knows those to whom one is known. For any set $A$ of $n$ citizens, there is some person among the other $n + 1$ who knows every one in $A$. Show that some citizen of the town knows all the others.

Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

Any citizen $a$ in this town is a member of some $n$-citizen set and so he is known to some citizen $b$ from the complement of this set. The set $\{a, b\}$ is a set of cardinality 2 and satisfy the property that “any two members of it know each other”.

Consider a set of citizens $B$ of maximum cardinality with the property that any two members of it know each other.

If $\text{card}(B) \leq n$ then $B$ is a subset of some $n$-citizen set and there is a person $b$ in the complement of that set who knows every member of $B$, and by hypothesis he is also known to the members of $B$. Now, the set $B \cup \{b\}$ has the property that “any two members of it know each other” and this contradicts the fact that $B$ was chosen to be of maximum cardinality. this contradiction proves that $\text{card}(B) \geq n + 1$.

Let $B_0$ be any subset of $B$ of cardinality $n + 1$. On one hand, $B_0$ has the property that “any two members of it know each other”, and on the other hand, there is a citizen $M$ in $B_0$, who knows any member of the complement of $B_0$, (since the complement has cardinality $n$,) Clearly $M$ knows all the citizens in the town.

\[\Box\]

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