Problem 11338. Proposed by Ovidiu Furdui, Cluj, Romania. Let $\Gamma$ denote the classical gamma function, and let $G(n) = \prod_{k=1}^{n} \Gamma(1/k)$. Find

$$\lim_{n \to \infty} \left( G(n+1)^{1/(n+1)} - G(n)^{1/n} \right).$$

Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria). The answer is $1/e$. In fact, with no extra effort one can prove the following lemma.

**Lemma.** Consider a function $f : [0, 1] \to \mathbb{R}$ such that $f(x) = 1 + \alpha x + \beta x^2 + o(x^2)$ in a neighbourhood of 0, and let $G_f(n) = \prod_{k=1}^{n} (kf(1/k))$. Then

$$G_f(n+1)^{1/(n+1)} - G_f(n)^{1/n} = \frac{1}{e} + \frac{1+2\alpha}{2en} + o\left(\frac{1}{n}\right).$$

**Proof.** Note first that

$$\ln f(1/k) = \frac{\alpha}{k} + \frac{2\beta - \alpha^2}{2k^2} + o\left(\frac{1}{k^2}\right).$$

and on the other hand, if $u_k = (k + \alpha + \frac{1}{2}) \ln k - k$, then

$$u_k - u_{k-1} = (k + \alpha + \frac{1}{2}) \ln k - (k + \alpha - \frac{1}{2}) \ln(k-1) - 1$$

$$= (k + \alpha + \frac{1}{2}) \ln k - (k + \alpha - \frac{1}{2}) \ln k + \ln(1 - \frac{1}{k}) - 1$$

$$= \ln k - 1 + \left(k + \alpha - \frac{1}{2}\right) \left(\frac{1}{k^2} + \frac{1}{2k^3} + o\left(\frac{1}{k^3}\right)\right)$$

$$= \ln k + \frac{\alpha}{k} + \frac{1+6\alpha}{12k^2} + o\left(\frac{1}{k^2}\right).$$

So that

$$\ln(kf(1/k)) - (u_k - u_{k-1}) = \frac{\delta}{k^2} + o\left(\frac{1}{k^2}\right) \quad \text{with} \quad \delta = \frac{12\beta - 1 - 6\alpha - 6\alpha^2}{12}.$$

From this we can draw two conclusions.

First, the convergence of the series $\sum_{k=1}^{\infty} (\ln(kf(1/k)) - (u_k - u_{k-1}))$, (with the convention $u_0 = 0$.)

Second, if $S$ denotes the sum of this series, then

$$S - \sum_{k=1}^{n} (\ln(kf(1/k)) - (u_k - u_{k-1})) = \sum_{k=n+1}^{\infty} (\ln(kf(1/k)) - (u_k - u_{k-1})) = \frac{\delta}{n} + o\left(\frac{1}{n}\right).$$

This is equivalent to $S - \ln G_f(n) + u_n = \frac{\delta}{n} + o\left(\frac{1}{n}\right)$, and consequently

$$\frac{1}{n} \ln G_f(n) = \ln n - 1 + \left(\alpha + \frac{1}{2}\right) \ln n + S - \frac{\delta}{n^2} + o\left(\frac{1}{n^2}\right),$$

and this proves that

$$\sqrt[n]{G_f(n)} = \frac{n}{e} \exp\left(\left(\alpha + \frac{1}{2}\right) \ln n + S - \frac{\delta}{n^2} + o\left(\frac{1}{n^2}\right)\right)$$

$$= \frac{n}{e} \left(1 + \left(\alpha + \frac{1}{2}\right) \ln n + S + \left(\alpha + \frac{1}{2}\right)^2 \frac{\ln^2 n}{2n^2} + \frac{S^2 - 2\delta}{2n^2} + S \left(\alpha + \frac{1}{2}\right) \frac{\ln n}{n^2}\right) + o\left(\frac{1}{n}\right)$$

and finally,

$$e \sqrt[n]{G_f(n)} = n + \frac{2\alpha + 1}{2} \ln n + S + \frac{(2\alpha + 1)^2}{8} \ln^2 n + S(2\alpha + 1) \frac{\ln n}{n} + S^2 - 2\delta \frac{\ln n}{2n} + o\left(\frac{1}{n}\right).$$

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Now, noting that $\frac{\ln^2(n+1)}{n+1} = \frac{\ln^2 n}{n} + o\left(\frac{1}{n}\right)$, $\frac{\ln(n+1)}{n+1} = \frac{\ln n}{n} + o\left(\frac{1}{n}\right)$, and $\frac{1}{n+1} = \frac{1}{n} + o\left(\frac{1}{n}\right)$, we conclude that

$$e^{\sqrt{G_f(n+1)}} - e^{\sqrt{G_f(n)}} = 1 + 2\alpha + \frac{1}{2} \cdot (\ln(n+1) - \ln n) + o\left(\frac{1}{n}\right) = 1 + \frac{2\alpha + 1}{2n} + o\left(\frac{1}{n}\right),$$

and this proves the lemma.

Now if we choose $f(x) = \Gamma(1 + x)$ and note that, in this case, $k f(1/k) = \Gamma(1/k)$, and that $f$ is analytic in the interval $]-1, +\infty[\,$, with $\alpha = \Gamma'(1) = -\gamma$, ($\gamma$ is the Euler constant,) we conclude that

$$e^{\sqrt{G(n+1)}} - e^{\sqrt{G(n)}} = \frac{1}{e} + \frac{1 - 2\gamma}{2\gamma e} + o\left(\frac{1}{n}\right).$$

In particular, this proves that the desired limit is $1/e$. □
Problem 11339. Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain. Let $F_n$ and $L_n$ denote the $n$th Fibonacci and Lucas numbers, respectively. Prove that for all $n \geq 1$,

$$\frac{1}{2} \left( \frac{F_{n+1}}{F_n} + \frac{L_{n+1}}{L_n} \right) \leq 2 - \frac{F_{n+1}}{F_{2n}}.$$

(The Fibonacci and Lucas numbers are given by the recurrence $a_{n+1} = a_n + a_{n-1}$, with $F_0 = 0$, $F_1 = 1$, $L_0 = 2$ and $L_1 = 1$.)

Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

Lemma. For $0 < x \leq 1$ we have $x + e^{-x \ln x} \leq 2$ with equality if, and only if, $x = 1$.

Proof. Let $f(x) = x + e^{-x \ln x}$ for $0 < x \leq 1$. Clearly, if $0 < x < 1$, we have,

$$f'(x) = 1 - (1 + \ln x)e^{-x \ln x} = (e^{x \ln x} - 1 - \ln x)e^{-x \ln x} \geq (1 + x \ln x - 1 - \ln x)e^{-x \ln x} = (1 - x)(-\ln x)e^{-x \ln x} > 0.$$

This proves that $f$ is strictly increasing on $[0, 1]$, and the lemma follows since $f(1) = 2$.

Taking $x = 1/F_n$ and $x = 1/L_n$ in the preceding inequality, we obtain

$$\frac{1}{F_n} + \frac{F_{n+1}}{F_n} \leq 2 \quad \text{and} \quad \frac{1}{L_n} + \frac{L_{n+1}}{L_n} \leq 2$$

with equality in both inequalities, if and only if, $n = 1$. It follows that

$$\frac{1}{2} \left( \frac{F_{n+1}}{F_n} + \frac{L_{n+1}}{L_n} \right) \leq 2 - \frac{F_{n+1}}{2L_n^2},$$

with equality, if and only if, $n = 1$.

Now, remembering that,

$$F_n = \frac{\phi^n - (-1/\phi)^n}{\sqrt{5}} \quad \text{and} \quad L_n = \phi^n + (-1/\phi)^n$$

with $\phi = \frac{\sqrt{5} + 1}{2}$, we conclude immediately that $F_{n+1} = \frac{F_n + L_n}{2}$ and $F_{2n} = F_nL_n$, and this proves the desired inequality, and that the equality occurs if and only if $n = 1$. \qed
Problem 11343. Proposed by David Beckwith, Sag Harbor, NY. Show that when \( n \) is a positive integer

\[
\sum_{k \geq 0} \binom{n}{k} \binom{2k}{k} = \sum_{k \geq 0} \binom{n}{2k} \binom{2k}{k} 3^{n-2k}.
\]

Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

Let \( A_n = \sum_{k \geq 0} \binom{n}{k} \binom{2k}{k} \). Since \( \binom{2k}{k} \leq 2^{2k} \) we conclude that \( A_n \leq \sum_{k=0}^{n} \binom{n}{k} 4^k = 5^n \). So, for \( |x| < 1/5 \),

\[
\sum_{n \geq 0} A_n x^n = \sum_{n \geq k \geq 0} \binom{n}{k} \binom{2k}{k} x^n = \sum_{k \geq 0} \binom{2k}{k} \left( \sum_{n \geq k} \binom{n}{k} x^n \right)
= \sum_{k \geq 0} \binom{2k}{k} \frac{x^k}{(1-x)^{k+1}} = \frac{1}{1-x} \cdot \frac{1}{\sqrt{1-4x}}
= \frac{1}{\sqrt{(1-x)(1-5x)}}
\]

(1)

Where we used the well-known expansions \( \sum_{n \geq q} \binom{n}{k} x^n = \frac{x^q}{(1-x)^{q+1}} \) and \( \sum_{k \geq 0} \binom{2k}{k} t^k = \frac{1}{\sqrt{1-4t}} \).

On the other hand, we note that \( (1-x)(1-5x) = 1 - 6x + 5x^2 = (1-3x)^2 - 4x^2 \) so, using the same expansions with different parameters, we obtain

\[
\frac{1}{\sqrt{(1-x)(1-5x)}} = \frac{1}{1-3x} \cdot \frac{1}{\sqrt{1-4x^2}} = \frac{1}{1-3x} \sum_{k \geq 0} \binom{2k}{k} \frac{x^{2k}}{(1-3x)^{2k}}
= \sum_{k \geq 0} 3^{-2k} \binom{2k}{k} \frac{(3x)^{2k}}{(1-3x)^{2k+1}} = \sum_{k \geq 0} 3^{-2k} \binom{2k}{k} \left( \sum_{n \geq 2k} \binom{n}{2k} (3x)^n \right)
= \sum_{n \geq 0} \left( \sum_{k \geq 0} \binom{n}{2k} \binom{2k}{k} 3^{n-2k} \right) x^n
\]

(2)

Now, comparing (1) and (2) proves the desired result. \( \Box \)

Remark. The same method proves the following more general identity,

\[
\sum_{k \geq 0} \binom{n}{k} \binom{2k}{k} z^k = \sum_{k \geq 0} \binom{n}{2k} \binom{2k}{k} z^{2k} (1+2z)^{n-2k}.
\]
Problem 11345. Proposed by Roger Cuculièr, France. Find all nondecreasing functions $f$ from $\mathbb{R}$ to $\mathbb{R}$ such that $f(x + f(y)) = f(f(x)) + f(y)$ for all real $x$ and $y$.

Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria). We will show that nondecreasing solutions $f : \mathbb{R} \rightarrow \mathbb{R}$ to the functional equation:
\begin{equation}
\forall (x, y) \in \mathbb{R}^2, \quad f(x + f(y)) = f(f(x)) + f(y),
\end{equation}
are:
\begin{itemize}
\item $\forall x$, \quad $f(x) = 0$.
\item $\forall x$, \quad $f(x) = x$.
\item $\forall x$, \quad $f(x) = \lambda \left\lceil \frac{x - \beta}{\lambda} \right\rceil$, for some $0 \leq \beta < \lambda$.
\item $\forall x$, \quad $f(x) = \lambda \left\lfloor \frac{x - \alpha}{\lambda} \right\rfloor$, for some $0 \leq \alpha < \lambda$.
\end{itemize}
In fact, it is straightforward and easy to check that these are solutions to the problem. Conversely, let’s consider a nondecreasing function $f : \mathbb{R} \rightarrow \mathbb{R}$ that satisfies the proposed functional equation, and let $G$ denote the direct image of $f$, that is $G = f(\mathbb{R})$.

- Setting $x = 0$ and $y = 0$ in (1) we obtain that $f(0) = 0$ and in particular $0 \in G$.
- Using this, and setting $x = 0$ in (1) we obtain $f(f(y)) = f(y)$ for every real $y$. That is $\forall g \in G$, $f(g) = g$.

Now, the functional equation (1) can be written in the following simpler form:
\begin{equation}
\forall (x, y) \in \mathbb{R} \times G, \quad f(x + y) = f(x) + f(y),
\end{equation}
- If for $g \in G$ we choose $x = -g$ in (2) we see that $f(-g) = -g$. This proves that $\forall g \in G$, $-g \in G$.
- Finally, if we choose $x = g', g' \in G$ in (2) we see that $g' + g = f(g' + g)$. This proves that $\forall (g, g') \in G^2$, $g + g' \in G$.

We have shown that $G$ is a subgroup of $(\mathbb{R}, +)$, so we distinguish the following three cases:

**case 1.** $G = 0$. Clearly this is equivalent to $f \equiv 0$.

**case 2.** $\overline{G} = \mathbb{R}$, i.e. $G$ is a dense subset of $\mathbb{R}$. Consider a real number $x$, and let $\epsilon$ be an arbitrary positive number. Using the density of $G$ in $\mathbb{R}$, we can find $g_1$ and $g_2$ in $G$ so that $x - \epsilon < g_1 < x < g_2 < x + \epsilon$. Now, since $f$ is nondecreasing we conclude that $f(g_1) \leq f(x) \leq f(g_2)$, but $f(g_1) = g_1$ and $f(g_2) = g_2$ so we have $x - \epsilon < f(x) < x + \epsilon$. This proves that $f(x) = x$ since $\epsilon$ is an arbitrary positive number.

**case 3.** $G = \lambda \mathbb{Z}$ where, by definition, $\lambda = \inf(G \cap \mathbb{R}^+ \setminus \{0\}) > 0$. Let $S$ denote the set $\{x \in \mathbb{R} : f(x) = \lambda\}$. Since $\lambda \in G$ we have $f(\lambda) = \lambda$ and consequently $S \neq \emptyset$. On the other hand, if $x \leq 0$ we have $f(x) \leq f(0) = 0 < \lambda$, and this proves that $S$ is bounded below by 0. So, we define $\beta = \inf S$, and we see easily that $0 \leq \beta \leq \lambda$, and that $f(\beta) \in [0, \lambda] \cap G = \{0, \lambda\}$.

- **case 3.1.** $f(\beta) = 0$ in particular $\beta < \lambda$. Using the fact that $f$ is nondecreasing we conclude that, for $0 \leq t < \beta$ we have $f(t) = 0$, and for $\beta < t \leq \lambda$ we have $f(t) = \lambda$, (since there is an element $y \in S$ such that $y < t$.) Using (2), we conclude from this that, for $\beta - \lambda < t \leq 0$, we have $\lambda = f(t + \lambda) = f(t) + \lambda$, that is $f(t) = 0$. So, we have proved that $f(t) = 0$ for every $t$ in the interval $[\beta - \lambda, \beta]$.

Now, Consider an element $x \in \mathbb{R}$, and define $k = \left\lceil \frac{x - \beta}{\lambda} \right\rceil$, then $k \geq \frac{x - \beta}{\lambda} > k - 1$ which is equivalent to $\beta - \lambda < x - \lambda k \leq \beta$, so that $f(x - \lambda k) = 0$. But using (2) and the fact that $\lambda k \in G$ we conclude that $f(x) = f(x - \lambda k + \lambda k) = f(x - \lambda k) + \lambda k = \lambda \left\lfloor \frac{x - \beta + \lambda}{\lambda} \right\rfloor$.

- **case 3.2.** $f(\beta) = \lambda$ in particular $0 < \beta$. Using the fact that $f$ is nondecreasing we conclude that, for $\beta \leq t \leq \lambda$ we have $f(t) = \lambda$, and for $0 \leq t < \beta$ we have $f(t) = 0$. (In fact, we have $0 \leq f(t) \leq f(\beta) = \lambda$, but $f(\beta) \neq \lambda$, by definition of $\beta$, so $f(t) \in G \cap [0, \lambda] = \{0\}$. Using (2), we conclude from this that, for $\beta - \lambda \leq t \leq 0$, we have $\lambda = f(t + \lambda) = f(t) + \lambda$, that is $f(t) = 0$. So, we have proved that $f(t) = 0$ for every $t$ in the interval $[\beta - \lambda, \beta]$.

Now, Consider an element $x \in \mathbb{R}$, and define $k = \left\lceil \frac{x - \beta + \lambda}{\lambda} \right\rceil$, then $k \leq \frac{x - \beta + \lambda}{\lambda} < k + 1$ which is equivalent to $\beta - \lambda < x - \lambda k < \beta$, so that $f(x - \lambda k) = 0$. But using (2) and the fact that $\lambda k \in G$ we conclude that $f(x) = f(x - \lambda k + \lambda k) = f(x - \lambda k) + \lambda k = \lambda \left\lceil \frac{x - \beta + \lambda}{\lambda} \right\rceil = \lambda \left\lfloor \frac{x + \alpha}{\lambda} \right\rfloor$, with $\alpha = \lambda - \beta$.

In conclusion, we have shown that $f$ must have one of the types considered in the beginning of the proof.
Problem 11347. Proposed by Mihaly Bencze, Brasov, Romania. Let $A = (x + y)/2$, $G = \sqrt{xy}$, and

\[ I = \frac{1}{e} \left( \frac{x^y}{y^x} \right)^{1/(x-y)}. \]

Determine all ordered 4-tuples $(\alpha, \beta, \gamma, \delta)$ of positive numbers with $\alpha > \beta$ and $\gamma > \delta$ such that for all distinct positive $x$ and $y$,

\[ I > \frac{\alpha A + \beta G}{\alpha + \beta} > (A^\gamma G^\delta)^{1/(\gamma + \delta)} > \sqrt{AG}. \]

Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

the answer is

\[ \left\{(\alpha, \beta, \gamma, \delta) \in (\mathbb{R}_+^*)^4 : 1 \leq \frac{\gamma}{\delta} \leq \frac{\alpha}{\beta} \leq 2 \right\}. \]

Clearly, the identric mean $I$ of $x$ and $y$ is positively homogeneous and symmetric, i.e.

\[ I(x, y) = I(y, x), \quad \text{and} \quad \forall \, \lambda > 0, \quad I(\lambda x, \lambda y) = \lambda I(x, y). \]

The same is true for the arithmetic mean $A$ and the geometric mean $G$ of $x$ and $y$, and consequently this must also be true for the means $\frac{\alpha A + \beta G}{\alpha + \beta}$, $(A^\gamma G^\delta)^{1/(\gamma + \delta)}$ and $\sqrt{AG}$.

This remark, proves that it is sufficient to consider the case $(x, y) = (e^t, e^{-t})$ with $t > 0$, and this is what we will do in the sequel.

For $(x, y) = (e^t, e^{-t})$ we have $I = I(t) = \exp\left(\frac{1}{\tanh t} - 1\right)$, $A = A(t) = \cosh t$ and $G = 1$. Let us consider successively the different inequalities.

1. $(A^\gamma G^\delta)^{1/(\gamma + \delta)} > \sqrt{AG}$. This is equivalent to $(\cosh t)^{\gamma/(\gamma + \delta)} > \sqrt{\cosh t}$, for $t > 0$, and since $\cosh t > 1$ for these values of $t$, we conclude that the preceding inequality is satisfied if, and only if, $\frac{\gamma}{\gamma + \delta} > \frac{1}{2}$, that is if, and only if, $\gamma > \delta$.

2. $\frac{\alpha A + \beta G}{\alpha + \beta} > (A^\gamma G^\delta)^{1/(\gamma + \delta)}$. This is equivalent to

\[ \forall \, t > 0, \quad \frac{\alpha \cosh t + \beta}{\alpha + \beta} > (\cosh t)^{\gamma/(\gamma + \delta)} \]

And if we denote $v$ for $\cosh t$ which is greater than 1 for $t > 0$, we conclude that the preceding inequality is satisfied if, and only if,

\[ \forall \, v > 1, \quad \frac{\alpha}{\alpha + \beta} > \frac{v^\nu - 1}{v - 1} \]

with $\nu = \frac{\gamma}{\gamma + \delta}$. Now, if this is true, then by letting $v$ tend to 1 we conclude that $\frac{\alpha}{\alpha + \beta} \geq \nu$.

Conversely, assume that $\frac{\alpha}{\alpha + \beta} \geq \nu$. since $\nu < 1$ we conclude that $s^{\nu - 1} < 1$ for $1 < s < v$, and by integration, we obtain that $v^\nu - 1 = \int_1^v s^{\nu - 1} \, ds < \nu(v - 1)$, that is

\[ \frac{v^\nu - 1}{v - 1} < \nu \leq \frac{\alpha}{\alpha + \beta}. \]

We conclude that the inequality $\frac{\alpha A + \beta G}{\alpha + \beta} > (A^\gamma G^\delta)^{1/(\gamma + \delta)}$ is satisfied if, and only if, $\nu \leq \frac{\alpha}{\alpha + \beta}$ which is equivalent to $\frac{\nu}{\gamma + \delta} \leq \frac{\alpha}{\alpha + \beta}$. 

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3. $I > \frac{\alpha A + \beta G}{\alpha + \beta}$. This is equivalent to

$$\forall t > 0, \quad \frac{t}{\tanh t} - 1 > \ln(1 + \mu(\cosh t - 1))$$

with $\mu = \frac{\alpha}{\alpha + \beta}$. Now, if $t$ is in the neighbourhood of 0 we have

$$\frac{t}{\tanh t} - 1 = \frac{1}{3} t^2 + O(t^4)$$

$$\ln(1 + \mu(\cosh t - 1)) = \frac{\mu}{2} t^2 + O(t^4)$$

So, if the proposed inequality is satisfied then we must have $\frac{1}{3} \geq \frac{\mu}{2}$ that is $\mu \leq \frac{2}{3}$.

Conversely, assume that $\mu = \frac{\alpha}{\alpha + \beta} \leq \frac{2}{3}$, and consider, for $t > 0$, the function

$$F(t) = \frac{t}{\tanh t} - 1 - \ln\left(\frac{1 + 2 \cosh t}{3}\right)$$

Clearly,

$$F'(t) = -\frac{1}{\tanh^2 t} - \frac{t}{\sinh^2 t} - \frac{2 \sinh t}{1 + 2 \cosh t} = \frac{1}{\sinh^2 t} \left(\frac{2 \sinh t + \frac{1}{3} \sinh(2t)}{1 + 2 \cosh t} - t\right)$$

Now, to find out the sign of $g(t)$ we calculate the derivative $g'(t)$. After doing some algebra, we obtain

$$g'(t) = \frac{2(\cosh t - 1) \sinh^2 t}{(1 + 2 \cosh t)^2}.$$ 

This proves that $g$ is increasing on the interval $[0, \infty]$, and since $g(0) = 0$ we conclude that $g(t) > 0$ for $t > 0$, hence $F$ is also increasing on the interval $[0, +\infty]$ and since $F(0) = 0$ we conclude that $F(t) > 0$ for $t > 0$. So, for $t > 0$ we have

$$\frac{t}{\tanh t} - 1 > \ln\left(1 + \frac{2}{3}(\cosh t - 1)\right) \geq \ln(1 + \mu(\cosh t - 1)).$$

Finally, the inequality $I > \frac{\alpha A + \beta G}{\alpha + \beta}$ is satisfied if, and only if, $\mu \leq \frac{2}{3}$ which is equivalent to $\frac{\alpha}{\beta} \leq 2$.

Conclusion, the set of all ordered 4-tuples $(\alpha, \beta, \gamma, \delta)$ of positive numbers such that for all distinct positive $x$ and $y$,

$$I > \frac{\alpha A + \beta G}{\alpha + \beta} > (A^7 G^6)^{1/(\gamma + \delta)} > \sqrt{AG}.$$ 

is

$$\left\{(\alpha, \beta, \gamma, \delta) \in (\mathbb{R}^*_+)^4 : 1 < \frac{\gamma}{\delta} \leq \frac{\alpha}{\beta} \leq 2\right\}.$$ 

This is the desired result.
Problem 11354. Proposed by Matthias Beck, San Francisco State University, San Francisco, CA, and Alexander Berkovich, University of Florida, Gainesville, FL. Find a polynomial \( f \) in two variables such that for all pairs \((s, t)\) of relatively prime positive integers,

\[
\sum_{m=1}^{s-1} \sum_{n=1}^{t-1} |mt - ns| = f(s, t).
\]

Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

The answer is

\[
f(s, t) = \frac{(t-1)(s-1)(2ts - t - s - 1)}{6}.
\]

Let \( \mathbb{N}_p \) denote the set of integers \( k \) satisfying \( 1 \leq k < p \). Now, Consider a pair \((s, t)\) of relatively prime positive integers, and denote \((m, n)\) be the set of integers \( p \) such that gcd\((m, n)\) is 1, and this proves that \( \phi(p) = 1 \), and this proves that \( s(t - q) \) since gcd\((s, t)\) = 1, and this proves that if \( s = q \) and \( q \) and \( p \) are both elements of \( \mathbb{N}_s \). We conclude that \( \varphi \) is also onto since the set \( \mathbb{N}_s \) is finite. This proves that \( \varphi \) is a permutation of the set \( \mathbb{N}_s \).

Now, \((m, n)\) \( \in \mathbb{A} \) if, and only if, \( q_m s + r_m - ns > 0 \) that is \( q_m = \lfloor mt/s \rfloor \) and \( mt = q_m s + r_m \). In fact, \( r_m \neq 0 \) since \( s \not| mt \), and this defines a mapping \( \varphi : \mathbb{N}_s \rightarrow \mathbb{N}_s \) by \( \varphi(m) = r_m \) which is one to one, (because \( r_q = r_q \) implies \( s(t - q) \) which, in turn, implies that \( s(t - q) \) since gcd\((s, t)\) = 1, and this proves that \( p = q \) since \( p \) and \( q \) are both elements of \( \mathbb{N}_s \).) We conclude that \( \varphi \) is also onto since the set \( \mathbb{N}_s \) is finite. This proves that \( \varphi \) is a permutation of the set \( \mathbb{N}_s \).

One checks easily that the mapping \((m, n) \mapsto (s - m, t - n)\) from \( \mathbb{A} \) to \( \mathbb{A} \) is one to one and onto, so that

\[
f(s, t) = \sum_{(m, n) \in \mathbb{A}} (mt - ns) - \sum_{(m, n) \in \mathbb{A}} (mt - ns) = \sum_{(m, n) \in \mathbb{A}} ((s - m)t - (t - n)s) - 2 \sum_{(m, n) \in \mathbb{A}} (mt - ns).
\]

For \( m \in \mathbb{N}_s \) let \( r_m \) and \( q_m \) be the rest and quotient of the euclidian division of \( mt \) by \( s \) respectively, that is \( q_m = \lfloor mt/s \rfloor \) and \( mt = q_m s + r_m \). In fact, \( r_m \neq 0 \) since \( s \not| mt \), and this defines a mapping \( \varphi : \mathbb{N}_s \rightarrow \mathbb{N}_s \) by \( \varphi(m) = r_m \) which is one to one, (because \( r_q = r_q \) implies \( s(t - q) \) which, in turn, implies that \( s(t - q) \) since gcd\((s, t)\) = 1, and this proves that \( p = q \) since \( p \) and \( q \) are both elements of \( \mathbb{N}_s \).) We conclude that \( \varphi \) is also onto since the set \( \mathbb{N}_s \) is finite. This proves that \( \varphi \) is a permutation of the set \( \mathbb{N}_s \).

Consequently,

\[
\sum_{(m, n) \in \mathbb{A}} (mt - ns) = \sum_{m=1}^{s-1} \sum_{n=1}^{t-1} (mt - ns) = \sum_{m=1}^{s-1} q_m \sum_{n=1}^{t-1} (mt - ns).
\]

So, taking the sum,

\[
f(s, t) = 2 \sum_{(m, n) \in \mathbb{A}} (mt - ns) = \sum_{m=1}^{s-1} q_m \sum_{n=1}^{t-1} (2mt - (q_m + 1 - n + n)s)
\]

\[
= \sum_{m=1}^{s-1} (2mt - (q_m + 1)s)q_m - \sum_{m=1}^{s-1} (mt + r_m - s)q_m
\]

\[
= \frac{1}{2} \sum_{m=1}^{s-1} (mt + r_m - s)(mt - r_m) = \frac{1}{2} \left( \sum_{m=1}^{s-1} (m^2 t^2 - r_m^2 - s(mt - r_m)) \right)
\]

\[
= \frac{1}{2} \left( (t^2 - 1) \sum_{m=1}^{s-1} m^2 - (t - 1) \sum_{m=1}^{s-1} m. \right)
\]

1
Where, in the final step we used the fact that $\varphi$ is a permutation to deduce that

$$\sum_{m=1}^{s-1} m^2 = \sum_{m=1}^{s-1} r_m^2 \quad \text{and} \quad \sum_{m=1}^{s-1} m = \sum_{m=1}^{s-1} r_m.$$

Finally,

$$f(s, t) = \frac{1}{s} \left( t^2 - 1 \right) \frac{(s - 1)s(2s - 1)}{6} - \frac{(t - 1)(s - 1)s}{2}$$

$$= \frac{(s - 1)(t - 1)}{6} \left( (t + 1)(2s - 1) - 3s \right)$$

$$= \frac{(s - 1)(t - 1)(2st - t - s - 1)}{6}.$$

And this is the proposed formula. \qed
**Problem 11355. Proposed by Jeffrey C. Lagarias, University of Michigan, Ann Arbor, MI.** Determine for which integers \( a \) the Diophantine equation

\[
\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{a}{xyz}
\]

has infinitely many integer solutions \((x, y, z)\) such that \( \gcd(a, xyz) = 1 \).

**Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria).**

We will show that the considered Diophantine equation has infinitely many integer solutions \((x, y, z)\) such that \( \gcd(a, xyz) = 1 \) if and only if \( a \) is odd.

1°. If \( a \) is even, then there is no integer solution \((x, y, z)\) to the considered Diophantine equation such that \( \gcd(a, xyz) = 1 \). In fact, if, on the contrary, we assume the existence of an integer solution \((x, y, z)\) to the considered Diophantine equation such that \( \gcd(a, xyz) = 1 \), then the fact that \( a \) and \( xyz \) are coprime proves that \( x, y \) and \( z \) are odd, and this implies that \( a = yz + zx + xy \) is also odd, which is a contradiction.

2°. If \( a = 2p + 1 \) for some integer \( p \), then there are infinitely many integer solutions \((x, y, z)\) to the considered Diophantine equation such that \( \gcd(a, xyz) = 1 \). In fact we will verify that the set \( \{(x_n, y_n, z_n) : n \in \mathbb{Z}\} \) defined by

\[
x_n = p + 1 + an, \quad y_n = -p - an, \quad \text{and} \quad z_n = a + (p + an)(p + 1 + an),
\]

is an infinite set of solutions to the considered Diophantine equation. First, since \( a \) is odd, we have \( \gcd(a, x_n) = \gcd(a, 2x_n) = \gcd(a, (2n + 1)a + 1) = \gcd(a, 1) = 1 \), and since \( y_n = x_n - (2n + 1)a \) we conclude that \( \gcd(a, y_n) = \gcd(a, x_n) = 1 \). Finally, since \( z_n = a - x_ny_n \) we have \( \gcd(a, z_n) = \gcd(a, x_ny_n) = 1 \). This discussion proves that \( \gcd(a, x_ny_nz_n) = 1 \).

On the other hand, it is obvious that \( x_nz_n + y_nz_n + x_ny_n = a \), and the verification is complete. \( \square \)
Problem 11356. Proposed by Michael Poghosyan, Yerevan State University, Yerevan, Armenia. Prove that for any positive integer \( n \),
\[
\sum_{k=0}^{n} \binom{n}{k}^2 \frac{(2k+1)(2n+2k)}{2k} = \frac{2^{4n} (n!)^4}{(2n)! (2n+1)!}.
\]

Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria). We will use generating functions. Recall that, for \( |x| < 1 \), we have
\[
\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-x^2)^n = \sum_{n=0}^{\infty} \frac{(2n)!}{(2n+1)!} x^{2n},
\]
and by integration, we obtain -for the same values of \( x \) - that,
\[
\arcsin(x) = \sum_{n=0}^{\infty} \frac{(2n)!}{(2n+1)!} x^{2n+1},
\]
So, if we consider the function \( f \) defined on the interval \((-1, 1)\) by \( f(x) = \frac{\arcsin(x)}{\sqrt{1-x^2}} \), we conclude that, for \(|x| < 1\), we have
\[
f(x) = \sum_{n=0}^{\infty} A_n x^{2n+1},
\]
with
\[
A_n = \sum_{k=0}^{n} \frac{\binom{2k}{k}}{(2k+1)2^{2k}} \frac{\binom{2n-2k}{n-k}}{2^{2n-2k}} = \frac{1}{2^{2n}} \sum_{k=0}^{n} \frac{\binom{2k}{k} \binom{2n-2k}{n-k}}{(2k+1)},
\]
and finally,
\[
A_n = \frac{(2n)!}{2^{2n}(n!)^2} \sum_{k=0}^{n} \binom{k}{2k} \binom{2n}{n-k} \binom{2n}{2k}.
\]
On the other hand, we note that, for \(|x| < 1\), we have
\[
\sqrt{1-x^2} \left( \frac{\arcsin(x)}{\sqrt{1-x^2}} \right)' = 1,
\]
which is equivalent to
\[
\forall x \in (-1, 1), \quad (1-x^2)f'(x) - xf(x) = 1.
\]
Using the power series expansion of \( f \), this implies that
\[
\sum_{n=1}^{\infty} (2n+1)A_n x^{2n} - \sum_{n=1}^{\infty} (2n-1)A_{n-1} x^{2n} = \sum_{n=1}^{\infty} A_{n-1} x^{2n} = 1,
\]
or
\[
A_0 + \sum_{n=1}^{\infty} ((2n+1)A_n - (2n)A_{n-1}) x^{2n} = 1.
\]
Therefore, the sequence \( (A_n)_{n \geq 0} \) satisfy the following recurrence relation
\[
\forall n \geq 1, \quad A_n = \frac{2n}{2n+1} A_{n-1} \quad \text{with} \quad A_0 = 1.
\]
This proves that for every nonnegative integer \( A_n = \frac{2^{2n}(n!)^2}{(2n+1)!} \). Now, comparing this to (1) proves the desired identity.

My solution was published in AMM February 2010, page 183.
Problem 11357. Proposed by Mehmet Şahin, Ankara, Turkey. Let \( I_a, I_b, I_c \) and \( r_a, r_b, r_c \) be respectively the excenters and exradii of the triangle \( ABC \). If \( \rho_a, \rho_b, \rho_c \) are the inradii of triangles \( I_aBC, I_bCA \) and \( I_cAB \), show that

\[
\frac{\rho_a}{r_a} + \frac{\rho_b}{r_b} + \frac{\rho_c}{r_c} = 1.
\]

Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

Notation: As usual, we will denote \( a, b \) and \( c \) respectively, the lengths of the sides \( BC, CA \) and \( AB \) of the triangle \( ABC \), and we will write \( p \) and \( S \) for the semi-perimeter and the area of \( ABC \). Finally, We will denote \( S_{XYZ} \) for the area of the triangle \( XYZ \).

Let \( \ell_a \) denote the perimeter of \( I_aBC \), then we can calculate the area \( S_{I_aBC} \) of the triangle \( I_aBC \) in two ways as follows : \( S_{I_aBC} = \frac{1}{2} \rho_a \ell_a = \frac{a}{2} \rho_a a \), therefore,

\[
\frac{\rho_a}{r_a} = \frac{a}{\ell_a}, \tag{1}
\]

In what follows, we will find an expression for \( \ell_a \).

Since

\[
S_{I_aCA} + S_{I_aAB} - S_{I_aCB} = S_{ABC} = S,
\]

we conclude that \( \frac{1}{2} \ell_a (b + c - a) = S \). So, by Heron’s formula, we get

\[
r_a = \frac{S}{p - a} = \sqrt{\frac{p(p - b)(p - c)}{p - a}}.
\]

On the other hand, if we denote \( A', B' \) and \( C' \) the points of tangency of the excircle of center \( I_a \) with the lines \( BC, CA \) and \( AB \) respectively, and define \( x \) and \( y \) by \( x = CA' = CB' \) and \( y = BA' = BC' \), then we have \( x + y = BA' + BA = a \) and \( y + c = AC' = AB' = b + x \). From these two equations we get \( x = p - b \) and \( y = p - c \).

Now, pythagoras theorem in the triangle \( I_aA'C \) allows us write,

\[
I_aC = \sqrt{r_a^2 + x^2} = \sqrt{\frac{p(p - b)(p - c)}{p - a} + (p - b)^2} = \sqrt{\frac{p(p - b)}{p - a}(p - c) + (p - a)(p - b)} = \sqrt{\frac{ab(p - b)}{p - a}}.
\]

Similarly, from the triangle \( I_aA'B \), we obtain

\[
I_aB = \sqrt{r_a^2 + y^2} = \sqrt{\frac{ac(p - c)}{p - a}},
\]

therefore,

\[
\ell_a = BC + I_aC + I_aB = a + \sqrt{\frac{ab(p - b)}{p - a}} + \sqrt{\frac{ac(p - c)}{p - a}} = \sqrt{\frac{a(p - a)}{p - a}}(\sqrt{a(p - a)} + \sqrt{b(p - b)} + \sqrt{c(p - c)})
\]

Using (1) we arrive at

\[
\frac{\rho_a}{r_a} = \frac{\sqrt{a(p - a)} + \sqrt{b(p - b)} + \sqrt{c(p - c)}}{\sqrt{a(p - a)} + \sqrt{b(p - b)} + \sqrt{c(p - c)}}, \tag{2}
\]
by symmetry, we obtain also

\[
\begin{align*}
\frac{\rho_b}{\tau_b} &= \frac{\sqrt{b(p-b)}}{\sqrt{a(p-a)} + \sqrt{b(p-b)} + \sqrt{c(p-c)}}. \\
\frac{\rho_c}{\tau_c} &= \frac{\sqrt{c(p-c)}}{\sqrt{a(p-a)} + \sqrt{b(p-b)} + \sqrt{c(p-c)}}.
\end{align*}
\]  

(3)  

(4)

and the result follows by summing (2), (3) and (4).
Problem 11360. Proposed by CEZAR LUPE, student, University of Bucharest, Bucharest, and TUDOREL LUPO, Decebal High School, Constanza, Romania. Let $f$ and $g$ be continuous real-valued functions on $[0, 1]$ satisfying the condition $\int_0^1 f(x)g(x)dx = 0$. Show that

$$\int_0^1 f^2 \int_0^1 g^2 \geq 4 \left( \int_0^1 f \int_0^1 g \right)^2$$

and

$$\int_0^1 f^2 \left( \int_0^1 f \right)^2 + \int_0^1 g^2 \left( \int_0^1 g \right)^2 \geq 4 \left( \int_0^1 f \int_0^1 g \right)^2.$$

Solution, by OMAR KOUBA (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

Let us start by proving the following lemma.

Lemma. Let $V$ denote a real inner product space, and consider three vectors $x$, $y$ and $u$ in $V$ satisfying

$$\langle x, u \rangle = \langle y, u \rangle = \langle u, u \rangle = 1 \quad \text{and} \quad \langle x, y \rangle = 0.$$ 

Then

$$4 \leq \|x\|^2 + \|y\|^2 \leq \|x\|^2\|y\|^2.$$ \hspace{1cm} (1)

Proof. For a real $\lambda$ we have

$$\|x + \lambda y - (1 + \lambda)u\|^2 = \|x\|^2 + \lambda^2\|y\|^2 + (1 + \lambda)^2\|u\|^2 + \lambda \langle x, y \rangle - 2(1 + \lambda) \langle x, u \rangle - 2\lambda(1 + \lambda) \langle y, u \rangle$$

$$= \|x\|^2 + \lambda^2\|y\|^2 + (1 + \lambda)^2 - 2(1 + \lambda) - 2\lambda(1 + \lambda)$$

$$= \|x\|^2 + \lambda^2\|y\|^2 - (1 + \lambda)^2.$$

Finally,

$$\forall \lambda \in \mathbb{R}, \quad \|x + \lambda y - (1 + \lambda)u\|^2 = (\|x\|^2 - 1) - 2\lambda + \lambda^2(\|y\|^2 - 1) \hspace{1cm} (2)$$

Since this second degree trinomial is non negative for every $\lambda \in \mathbb{R}$, we conclude that its discriminant is nonpositive, i.e. $1 - (\|x\|^2 - 1)(\|y\|^2 - 1) \leq 0$, and this is the second inequality: $\|x\|^2 + \|y\|^2 \leq \|x\|^2\|y\|^2$.

On the other hand, choosing $\lambda = 1$ in (2) we find that $\|x\|^2 + \|y\|^2 - 4 \geq 0$ which is the first inequality. This ends the proof of the lemma.

Corollary. Let $V$ denote a real inner product space, and let $x$, $y$ and $u$ be vectors in $V$ satisfying $\langle x, y \rangle = 0$ and $\|u\| = 1$. Then the following inequality is true,

$$4 \langle (x, y) \rangle \langle (y, u) \rangle \leq \langle (y, u) \rangle^2\|x\|^2 + \langle (x, u) \rangle^2\|y\|^2 \leq \|x\|^2\|y\|^2.$$ \hspace{1cm} (3)

In fact, this inequality is clearly true if $\langle x, u \rangle = 0$ or $\langle y, u \rangle = 0$. On the other hand, if $\langle x, u \rangle \neq 0$ and $\langle y, u \rangle \neq 0$, we apply (1) to $x' = \frac{1}{\langle x, u \rangle} x$, $y' = \frac{1}{\langle y, u \rangle} y$ and $u$, to get (3).

Now, if we consider $V$ to be the space $C([0, 1])$ of real continuous functions on $[0, 1]$, equipped with the inner product $\langle F, G \rangle = \int_0^1 FG$, and if we choose the element $1 \in V$ to be the constant function which is equal to 1 on $[0, 1]$, then the statement of the problem follows from the corollary applied to this setting with $x$, $y$ and $u$ equal to $f$, $g$ and $1$ respectively.
Problem 11361. Proposed by Finbarr Holland, University Cork College, Cork, Ireland. The Lemoine point of a triangle is the unique point \( L \) inside the triangle such that the distances from \( L \) to the sides are proportional to the corresponding side lengths.

Given a circle \( G \) and distinct fixed points \( B, C \) on \( G \), let \( K \) be the locus of the Lemoine point of \( ABC \) as \( A \) traverses the circle. Show that \( K \) is an ellipse.

Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

As usual, we will denote \( a, b \) and \( c \) respectively, the lengths of the sides \( BC, CA \) and \( AB \) of the triangle \( ABC \).

Let us start by proving the following Lemma.

**Lemma.** The Lemoine point \( L \) of a triangle \( ABC \) is the barycenter of the weighted points \((A; a^2), (B; b^2)\) and \((C; c^2)\), i.e.

\[
a^2\overrightarrow{LA} + b^2\overrightarrow{LB} + c^2\overrightarrow{LC} = 0. \tag{1}
\]

**Proof.** We will write \( S_{XYZ} \) for the area of the triangle \( XYZ \). Let \( \ell_a, \ell_b \) and \( \ell_c \) denote the distances of the Lemoine point to the sides \( BC, CA \) and \( AB \) respectively. By definition, there is a positive constant \( \kappa \) such that \( \ell_a = 2\kappa a, \ell_b = 2\kappa b \) and \( \ell_c = 2\kappa c \). It follows that \( S_{LBC} = \kappa a^2, S_{LCA} = \kappa b^2 \) and \( S_{LAB} = \kappa c^2 \). This ends the proof of the lemma, using the following classical result.

If \( P \) is a point inside a triangle \( ABC \), then it is the barycenter of the weighted points \((A; \lambda), \( (B; \mu) \) and \( \( C; \nu) \) for some positive real numbers \( \lambda, \mu \) and \( \nu \) such that \( \lambda + \mu + \nu = 1 \). To determine \( \lambda, \mu \) and \( \nu \) we note that the equality \( \lambda\vec{PA} + \mu\vec{PB} + \nu\vec{PC} = 0 \), implies

\[
\lambda \det(\vec{PA}, \vec{PB}) + \nu \det(\vec{PC}, \vec{PB}) = 0 \quad \text{and} \quad \lambda \det(\vec{PA}, \vec{PC}) + \mu \det(\vec{PB}, \vec{PC}) = 0.
\]

Consequently,

\[
\lambda S_{PAB} = \nu S_{PBC} \quad \text{and} \quad \lambda S_{PCA} = \mu S_{PBC},
\]

therefore,

\[
\frac{S_{PBC}}{\lambda} = \frac{S_{PCA}}{\mu} = \frac{S_{PAB}}{\nu}.
\]

So \( P \) is the barycenter of the weighted points \((A; S_{PBC}), (B; S_{PCA}) \) and \((C; S_{PAB})\).

Let us come to our problem. Without loss of generality, we may consider cartesian coordinate axes so that the points \( B, C \) and the center \( \Gamma \) of \( G \) have cartesian coordinates \((-1, 0), (0, 1) \) and \((0, t) \) respectively. Then the coordinates \((\alpha, \beta)\) of \( A \in G \) must satisfy, the condition \( \alpha^2 + (\beta - t)^2 = \Gamma A^2 = 1 + t^2 \), that is

\[
\alpha^2 + \beta^2 - 2t\beta - 1 = 0. \tag{2}
\]

Now, we have, \( a^2 = BC^2 = 4 \), and

\[
b^2 = AC^2 = (\alpha - 1)^2 + \beta^2 = 2 - 2\alpha + 2t\beta \\
c^2 = AB^2 = (\alpha + 1)^2 + \beta^2 = 2 + 2\alpha + 2t\beta.
\]

so that \( a^2 + b^2 + c^2 = 8 + 4t\beta \).

It follows that

\[
\overrightarrow{OL} = \frac{c^2 - b^2}{a^2 + b^2 + c^2} \overrightarrow{OC} + \frac{a^2}{a^2 + b^2 + c^2} \overrightarrow{OA} = \frac{\alpha}{2 + t\beta} \overrightarrow{OC} + \frac{1}{2 + t\beta} \overrightarrow{OA},
\]

therefore, the coordinates \((x, y)\) of \( L \) satisfy

\[
x = \frac{2\alpha}{2 + t\beta} \quad \text{and} \quad y = \frac{\beta}{2 + t\beta},
\]
or equivalently,
\[ \alpha = \frac{x}{1 - ty} \quad \text{and} \quad \beta = \frac{2y}{1 - ty}, \]

So, using (2) we conclude that \( L \in K \) if, and only if, the coordinates \((x, y)\) of \( L \) satisfy
\[
\left( \frac{x}{1 - ty} \right)^2 + \left( \frac{2y}{1 - ty} \right)^2 - \frac{4ty}{1 - ty} - 1 = 0
\]
which is equivalent to
\[
x^2 + (4 + 3t^2)y^2 - 2ty - 1 = 0
\]
or
\[
x^2 + (4 + 3t^2)\left( y - \frac{t}{4 + 3t^2} \right)^2 = \frac{4 + 4t^2}{4 + 3t^2}
\]
This is clearly the cartesian equation of an ellipse, which is the desired result.

Remark: Consider two distinct points \( B \) and \( C \) in the plane \( \mathcal{P} \), and define the “Lemoine mapping” \( \mathcal{L} \) as follows: the image \( \mathcal{L}(A) \) of a point \( A \in \mathcal{P} \) is the Lemoine point of the triangle \( ABC \). One can show the following two results concerning this mapping:

1°. The image \( \mathcal{E} = \mathcal{L}(\mathcal{P}) \) of the whole plane \( \mathcal{P} \) is the compact set bounded by the ellipse whose foci are \( B \) and \( C \) and the length of its largest axis is equal to \( \frac{1}{2}BC \).

2°. The image \( \mathcal{L}(C) \) of any circle \( C \) in the plane is an ellipse. Of course, this generalizes the statement of the problem. The proof makes use of 1°. and a similar but a little bit more complicated calculations as in the solution of the problem.

My solution was published in AMM February 2010, page 186.
Problem 11369. Proposed by DONALD KNUTH, Stanford University, Stanford, CA. Prove that for all real 
t, and all \( \alpha \geq 2 \),
\[ e^{\alpha t} + e^{-\alpha t} - 2 \leq (e^t + e^{-t})^\alpha - 2^\alpha \]

Solution, by OMRAIN KOUBA (Higher Institute for Applied Sciences and Technology, Damascus, Syria).
For \( \alpha = 2 \) the inequality becomes an equality, and there is nothing to be proved, so we will assume that 
\( \alpha > 2 \).
Let us consider the function 
\[ F : \mathbb{R} \rightarrow \mathbb{R}, \quad F(t) = (e^t + e^{-t})^\alpha - (e^{\alpha t} + e^{-\alpha t}) + 2 - 2^\alpha \]

Clearly, \( F \) is an even function so we only need to prove that \( F(t) \geq 0 \) for \( t \geq 0 \). But we have
\[
F'(t) = \alpha \left( (e^t - e^{-t}) (e^t + e^{-t})^{\alpha - 1} - (e^{\alpha t} - e^{-\alpha t}) \right) \\
= \frac{\alpha}{(e^t + e^{-t})^\alpha} \left( \tanh(t) - \left( \left( \frac{e^t}{e^t + e^{-t}} \right)^\alpha - \left( \frac{e^{-t}}{e^t + e^{-t}} \right)^\alpha \right) \right) \\
= \frac{\alpha}{(e^t + e^{-t})^\alpha} \left( \tanh(t) - \left( \left( \frac{1 + \tanh(t)}{2} \right)^\alpha - \left( \frac{1 - \tanh(t)}{2} \right)^\alpha \right) \right) \\
= - \frac{\alpha}{2^\alpha (e^t + e^{-t})^\alpha} G(\tanh(t))
\]

with
\[ G(x) = (1 + x)^\alpha - (1 - x)^\alpha - 2^\alpha x. \]

Now, since \( \alpha > 2 \) we have \( (1 - x)\alpha - 2 \leq 1 \leq (1 + x)\alpha - 2 \) for \( x \in [0,1] \). It follows that \( G'' \geq 0 \) on the interval 
[0,1], and consequently \( G \) is convex on the interval [0,1]. But \( G(0) = 0 \) and \( G(1) = 0 \), so the convexity of 
\( G \) implies that \( G(x) \leq 0 \) for every \( x \in [0,1] \). Hence, \( F'(t) \geq 0 \) for \( t \geq 0 \). It follows that the function \( F \) is 
nondecreasing on the interval [0, +\( \infty \)], and satisfies \( F(0) = 0 \), therefore \( F(t) \geq 0 \) for \( t \geq 0 \) and the desired 
inequality follows.
Problem 11371. Proposed by Ovidiu Furdui, University of Toledo, OH. Let $A$ denote the Glaisher-Kinkelin constant, given by

$$A = \lim_{n \to \infty} n^{-n^2/2-n/2-1/12} e^{n^2/4} \prod_{k=1}^{n} k^k = 1.2824 \ldots$$

Evaluate in closed form

$$A^6 \prod_{n=1}^{\infty} \left(e^{-1}(1 + 1/n)^n\right)^{(-1)^n}.$$ 

Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

The answer is: $e \sqrt{\pi} \sqrt{2}$.

Let us denote $P_m$ the finite product $\prod_{n=1}^{m} \left(e^{-1}(1 + 1/n)^n\right)^{(-1)^n}$. Clearly we have

$$P_{2m} = \prod_{k=1}^{m} \frac{e^{-1}(1 + \frac{1}{2k})^{2k}}{\prod_{k=1}^{m} \left(1 + \frac{1}{2k-1}\right)^{2k-1}}$$

$$= \prod_{k=1}^{m} \frac{(2k+1)^{2k}}{(2k-1)^{2k-1}} / \prod_{k=1}^{m} \frac{(2k)^{2k-1}}{(2k-1)^{2k-1}}$$

$$= \left(\prod_{k=1}^{m} (2k+1)^{2k} \prod_{k=1}^{m} (2k-1)^{2k-1}\right) / \left(\prod_{k=1}^{m} (2k)^{2k} \prod_{k=1}^{m} (2k-1)^{2k-1}\right)$$

$$= (2m+1)^{2m} \prod_{k=1}^{m} \frac{(2k+1)^{2k-1}}{(2k-1)^{2k-1}} / \prod_{k=1}^{m} \frac{(2k)^{2k}}{(2k-1)^{2k-1}}$$

$$= (2m+1)^{2m} \frac{2 \cdot 4 \cdots (2m)}{1 \cdot 3 \cdots (2m-1)} \left(\prod_{k=1}^{m} (2k-1)^{2k-1}\right)^2 / \left(\prod_{k=1}^{m} (2k)^{2k}\right)^4$$

$$= (2m+1)^{2m} \frac{2^{2m}(m!)^2}{(2m)!} \left(\prod_{k=1}^{m} k^k\right)^2 / \left(\prod_{k=1}^{m} (2k)^{2k}\right)^4$$

But

$$\prod_{k=1}^{m} (2k)^{2k} = \prod_{k=1}^{m} 2^{2k} \left(\prod_{k=1}^{m} k^k\right)^2 = 2^{m(m+1)} G_m^2$$

where we denoted $G_m$ for the product $\prod_{k=1}^{m} k^k$. Finally, we conclude that

$$P_{2m} = (2m+1)^{2m} \frac{2^{2m}(m!)^2}{(2m)!} \frac{G_m^2}{2^{m(m+1)} G_m^2}.$$ 

Now, in the neighborhood of $+\infty$, we have $G_m \sim Am^{m^2/2+m/2+1/12} e^{-m^2/4}$ and also $m! \sim \sqrt{2\pi mM} e^{-m}$ by Stirling’s formula, so that, after simplification, we obtain $P_{2m} \sim \frac{e^{\sqrt{\pi} \sqrt{2}}}{A^6} (1 + \frac{1}{2m})^{2m}$. We conclude that

$$\lim_{m \to \infty} P_{2m} = \frac{e^{\sqrt{\pi} \sqrt{2}}}{A^6}$$

On the other hand, $P_{2m+1} = P_{2m} \times \frac{e}{(1 + 1/(2m+1))^{2m+1}}$ so we have also

$$\lim_{m \to \infty} P_{2m+1} = \frac{e^{\sqrt{\pi} \sqrt{2}}}{A^6}$$

and consequently, $\lim_{n \to \infty} A^6 P_n = e^{\sqrt{\pi} \sqrt{2}}$, which is the desired conclusion. 

\[\square\]
Problem 11374. Proposed by HARLEY FLANDERS and HUGH L. MONTGOMERY, University of Michigan, Ann Arbor, MI. Let $a, b, c$ and $m$ be positive integers such that $abcm = 1 + a^2 + b^2 + c^2$. Show that $m = 4$.

Solution, by OMARAN KOUBA (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

For a positive integer $m$ let $(E_m)$ denote the Diophantine equation $abcm = 1 + a^2 + b^2 + c^2$, and let $S_m$ denote the set of its solutions $(a, b, c)$ in positive integers.

• First, we will show that if $S_m$ is not empty then $m$ is multiple of 4. In fact, let us consider a solution $(a, b, c)$ from $S_m$. If one of the numbers $a, b$ or $c$ is even, then $a^2 + b^2 + c^2 = abcm - 1$ must be odd, and consequently, exactly one of $a, b$ and $c$ is odd and the other two are even, this implies that $a^2 + b^2 + c^2 = 1 \mod 4$ and $abcm - 1 = -1 \mod 4$ which is a contradiction. It follows that $a, b$ and $c$ are all odd, so $1 + a^2 + b^2 + c^2$ is a multiple of 4. Now, $4 | m(abc)$ and $\gcd(4, abc) = 1$ so $4 | m$, and this is what we claimed.

• Now, let us assume that $S_m$ is not empty for some $m = 4\mu$ with $\mu \geq 2$. It follows that the set

$$N = \{ a + b + c : a \leq b \leq c, \text{ and } (a, b, c) \in S_{4\mu} \}$$

is a non-empty subset of positive integers, and consequently, it contains a smallest element. So, let $(\alpha, \beta, \gamma)$ be the element from $S_{4\mu}$ that satisfy $\alpha \leq \beta \leq \gamma$ and $\alpha + \beta + \gamma = \min N$.

Let us consider the second degree polynomial

$$P(X) = X^2 - 4\mu \alpha \beta X + 1 + a^2 + \beta^2,$$

Clearly, $P(\gamma) = 0$, so let $\gamma'$ be the second zero of $P(X)$. From $\gamma \gamma' = 1 + a^2 + \beta^2 > 0$ and $\gamma + \gamma' = 4\mu \beta$ we conclude that $\gamma'$ is a positive integer. On the other hand $P(\beta) = 1 + a^2 - 2(2\mu a - 1)\beta^2$, with

$$2(2\mu a - 1)\beta^2 > 2(2\alpha - 1)\beta^2 \geq 2\alpha \beta^2 \geq 2\alpha^2 \geq a^2 + 1,$$

therefore $P(\beta) < 0$, and consequently $\gamma' < \beta < \gamma$. This is a contradiction since $(\alpha, \beta, \gamma') \in S_{4\mu}$ and $\alpha + \beta + \gamma' < \alpha + \beta + \gamma$.

So we have proved that $S_m \neq \emptyset \implies m = 4$, which is the desired result.

Remark. One can develop the idea of the proof to find infinitely many solutions to $E_4$, and even to generate all the solutions in $S_4$. For instance, if we define $(A_n)_{n \geq 0}$ by $A_0 = A_1 = A_2 = 1$ and $A_{n+1} = 4A_n A_{n-1} - A_{n-2}$ then

$$\{(A_k, A_{k+1}, A_{k+2}) : k \geq 0\} \subset S_4.$$
Problem 11375. Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania. The Brocard point of a triangle $ABC$ is the interior point $\Omega$ for which the angles $\angle B\Omega C$, $\angle C\Omega A$, and $\angle A\Omega B$ have the same radian measure. Let $\omega$ be that measure. Regarding the triangle as a figure in the Euclidean plane $\mathbb{R}^2$, show that if the vertices belong to $\mathbb{Z} \times \mathbb{Z}$, then $\omega/\pi$ is irrational.

Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

Notation. As usual, we will denote $a$, $b$, and $c$ respectively, the lengths of the sides $BC$, $CA$ and $AB$ of the triangle $ABC$ and $\hat{A}$, $\hat{B}$ and $\hat{C}$ the measures of the angles of the same triangle. Finally $R$ represents its circumradius.

In the triangle $\Omega BC$ we have $\angle \Omega BC = \omega$, $\angle \Omega CB = \hat{C} - \omega$ and $\angle B\Omega C = \pi - \hat{C}$, so by the sine rule we have $\frac{\Omega C}{\sin \omega} = \frac{a}{\sin \hat{C}}$. Therefore, $\Omega C = \frac{a}{c} 2R \sin \omega$, and similarly $\Omega B = \frac{c}{b} 2R \sin \omega$.

Now, Using the cosine rule in triangles $\Omega BC$ we obtain
\[
a^2 = \Omega B^2 + \Omega C^2 - 2\Omega B \Omega C \cos(\angle B\Omega C)
= \left(\frac{c^2}{b^2} + \frac{a^2}{c^2} + \frac{2a}{b} \cos \hat{C}\right) 4R^2 \sin^2 \omega
\]
but $2ab \cos \hat{C} = a^2 + b^2 - c^2$, so that
\[
a^2 = \left(\frac{a^2}{b^2} + \frac{a^2}{c^2} + 1\right) 4R^2 \sin^2 \omega
\]
that is
\[
\sin^2 \omega = \frac{a^2 b^2 c^2}{4R^2 (a^2 b^2 + b^2 c^2 + c^2 a^2)}.
\]
But, $\frac{abc}{2R}$ is twice the area of the triangle $ABC$, so, by Heron’s formula we can write
\[
\frac{a^2 b^2 c^2}{4R^2} = \frac{1}{4} (a + b + c)(a + b - c)(a - b + c)(-a + b + c)
= \frac{1}{4} (2(a^2 b^2 + b^2 c^2 + a^2 c^2) - a^4 - b^4 - c^4)
\]
So (1) becomes
\[
\sin^2 \omega = \frac{2(a^2 b^2 + b^2 c^2 + a^2 c^2) - a^4 - b^4 - c^4}{4(a^2 b^2 + b^2 c^2 + c^2 a^2)}
\]
and consequently,
\[
\cos(2\omega) = 1 - 2 \sin^2 \omega = \frac{a^4 + b^4 + c^4}{2(a^2 b^2 + b^2 c^2 + c^2 a^2)}
\]
Noting that
\[
2(a^4 + b^4 + c^4 - a^2 b^2 - b^2 c^2 - c^2 a^2) = (a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2 \geq 0
\]
we conclude from (2) that $\cos(2\omega) \geq 1/2$ with equality if, and only if, $a = b = c$.

Now, what conclusions can we draw from the fact that the vertices of $ABC$ belong to $\mathbb{Z} \times \mathbb{Z}$? First, $ABC$ cannot be equilateral, for if $A(k, \ell)$ and $B(n, m)$ are two points from $\mathbb{Z} \times \mathbb{Z}$ and if $ABC$ is equilateral, then the coordinates of $C$ are $(\frac{1}{2} (k + m - \sqrt{3}(n - \ell)), \frac{1}{2} (\ell + n + \sqrt{3}(m - k)))$, with $\epsilon \in \{-1, +1\}$, and consequently $C \notin \mathbb{Z} \times \mathbb{Z}$. Therefore, $\cos(2\omega) > \frac{1}{2}$. Moreover, if the vertices of $ABC$ belong to $\mathbb{Z} \times \mathbb{Z}$ then $a^2$, $b^2$, and $c^2$ are positive integers, and by (2), $\cos(2\omega)$ is a rational number. So if the vertices of $ABC$ belong to $\mathbb{Z} \times \mathbb{Z}$ then $\cos(2\omega)$ is a rational number from the open interval $(\frac{1}{2}, 1)$. 

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Now the conclusion of the problem follows from the following general lemma:

**Lemma.** If \( r \) and \( \cos(\pi r) \) are rational numbers then \( \cos(\pi r) \in \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\} \).

**Proof.** Let us assume that \( r = \frac{m}{n} \) with \( \gcd(n, m) = 1 \), then \( \xi = e^{i\pi r} \) is a primitive \( n \)th root of unity, and the monic irreducible polynomial of \( \xi \) over \( \mathbb{Q} \) is, in fact, the cyclotomic polynomial \( \Phi_n(X) \) which is of degree \( \varphi(n) \), (\( \varphi \) is the Euler totient function.) On the other hand, it follows from \( \cos(\pi r) = q \in \mathbb{Q} \) that \( \xi \) is a root of the polynomial \( X^2 - 2qX + 1 \) from \( \mathbb{Q}[X] \), so we must have \( \varphi(n) \leq 2 \), which implies that \( n \in \{1, 2, 3, 4, 6\} \). Clearly if \( \xi \) is an \( n \)th root of unity with \( n \in \{1, 2, 3, 4, 6\} \) then \( \cos(\pi r) = \Re(\xi) \in \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\} \). □
Problem 11376. Proposed by Bogdan M. Baishanski, The Ohio State University, Columbus OH. Given a real number \(a\) and a positive integer \(n\), let
\[
S_n(a) = \sum_{a_0 < k \leq (a+1)n} \frac{1}{\sqrt{kn - an}}.
\]

For which \(a\) does the sequence \(\langle S_n(a) \rangle\) converge?

Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

The answer is that the sequence \(\langle S_n(a) \rangle\) converges if, and only if, \(a\) is a rational number. Moreover, for \(a \in \mathbb{Q}\) we have \(\lim_{n \to \infty} S_n(a) = 2\).

In fact, let us note that,
\[
S_n(a) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n+\lfloor an \rfloor} \frac{1}{\sqrt{k-an}} = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{1}{\sqrt{k+\lfloor an \rfloor-an}} = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{1}{\sqrt{k-\{an\}}},
\]
where \(\{x\}\) denotes the fractional part of \(x\). We conclude that \(S_n(a) = \frac{1}{\sqrt{n(1-\{an\})}} + R_n(a)\) with
\[
R_n(a) = \frac{1}{\sqrt{n}} \sum_{k=2}^{n} \frac{1}{\sqrt{k-\{an\}}},
\]
but, for \(2 \leq k \leq n\) we have
\[
2(\sqrt{k+1} - \sqrt{k}) \leq \frac{1}{\sqrt{k}} \leq \frac{1}{\sqrt{k-\{an\}}} \leq \frac{1}{\sqrt{k-1}} \leq 2(\sqrt{k-1} - \sqrt{k-2})
\]
and consequently,
\[
2\frac{\sqrt{n+1} - \sqrt{2}}{\sqrt{n}} \leq R_n(a) \leq 2\frac{\sqrt{n} - 1}{\sqrt{n}}.
\]
This proves that for every real number \(a\) we have \(\lim_{n \to \infty} R_n(a) = 2\), and hence, the sequence \(\langle S_n(a) \rangle\) converges if, and only if, the sequence \(\langle \frac{1}{\sqrt{n(1-\{an\})}} \rangle\) converges. So, let us consider the following two cases:

1°. \(a \in \mathbb{Q}\). We may assume that \(a = \frac{p}{q}\) for some integer \(p\) and some positive integer \(q\) with \(\gcd(p, q) = 1\).

Then, for every positive integer \(n\), the fractional part of \(\{na\}\) is one of the fractions \(\{0, \frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q}\}\) and consequently \(1 - \{na\} \geq \frac{1}{q}\). It follows that \(\lim_{n \to \infty} n(1-\{na\}) = +\infty\). We conclude that, if \(a \in \mathbb{Q}\) then \(\lim_{n \to \infty} S_n(a) = 2\).

2°. \(a \notin \mathbb{Q}\). We consider the development of \(a\) as a simple continued fraction \(a = [x_0, x_1, \ldots, x_n, \ldots]\), and, as usual, we define the sequence of convergents \(\langle \frac{p_n}{q_n} \rangle\) by \(\frac{p_n}{q_n} = [x_0, x_1, \ldots, x_n]\). Then, it is well-known from the theory of continued fractions that, for each non-negative integer \(n\), one has
\[
0 < a - \frac{p_{2n}}{q_{2n}} < \frac{1}{q_{2n}^2} \quad \text{and} \quad 0 < \frac{p_{2n+1}}{q_{2n+1}} - a < \frac{1}{q_{2n+1}^2}.
\]
It follows that, \(p_{2n} = \lfloor q_{2n}a\rfloor\) and \(p_{2n+1} = 1 + \lfloor q_{2n+1}a\rfloor\), hence,
\[
q_{2n} - 1 < q_{2n}(1-\{q_{2n}a\}) \quad \text{and} \quad q_{2n+1}(1-\{q_{2n+1}a\}) < 1.
\]
This proves that \(\liminf_{n \to \infty} \frac{1}{n(1-\{na\})} = 0\) and \(\limsup_{n \to \infty} \frac{1}{n(1-\{na\})} \geq 1\) since \(\langle q_n \rangle\) is a strictly increasing sequence of positive integers. Therefore, if \(a \notin \mathbb{Q}\) then the sequence \(\langle \frac{1}{n(1-\{an\})} \rangle\) diverges, and so does the sequence \(\langle S_n(a) \rangle\).

This completes the proof. \(\square\)
Problem 11377. Proposed by Christopher Hillar, Texas A&M University, College Station, TX, and Lionel Levine, Massachusetts Institute of Technology, Cambridge, MA. Given a monic polynomial $p$ of degree $n$ with complex coefficients, let $A_p$ be the $(n+1) \times (n+1)$ matrix with $p(-i+j)$ in position $(i,j)$, and let $D_p$ be the determinant of $A_p$. Show that $D_p$ depends only on $n$, and find its value in terms of $n$.

Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria). The answer is: $D_p = (n!)^{n+1}$.

The main ingredient is the following lemma:

**Lemma.** Let $\mathbb{C}_n[X]$ denote the vector space of polynomials of degree at most $n$ with complex coefficients. Then

$$\forall Q \in \mathbb{C}_n[X], \quad \Delta^n(Q)(X) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} Q(X + k) = Q^{(n)}(X).$$

Where $\Delta$ is the difference operator defined by $\Delta(Q)(X) = Q(X+1) - Q(X)$. (Note that $Q^{(n)}(X)$ is constant for $Q$ in $\mathbb{C}_n[X]$.)

Now, consider a polynomial $p$ of degree $n$, not necessarily monic. If $C_j = [p(j-1), \ldots, p(j-n-1)]^T$ denotes the $j$th column of the matrix $A_p$, then the determinant $D_p$ will not change if we replace the first column $C_1$ by $C_1' = C_1 + \sum_{k=1}^{n} \binom{n}{k} (-1)^k C_{k+1}$. Using the lemma, and the fact that $p^{(n)}$ is constant since deg $p = n$, we see that $C_1' = (-1)^n p^{(n)} [1, 1, \ldots, 1]^T$, so that

$$D_p = (-1)^n p^{(n)} \det \begin{pmatrix} 1 & p(1) & \cdots & p(n) \\ 1 & p(0) & \cdots & p(n-1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & p(1-n) & \cdots & p(0) \end{pmatrix}.$$

Let $A_p'$ be the $(n+1) \times (n+1)$ matrix, whose $i$th row is given by $R_i = [1, p(2-i), \ldots, p(n+1-i)]$. The determinant $\det(A_p')$ will not change if we successively replace $R_i$ by $R_i - R_{i+1}$ with $i$ varying from 1 to $n$. We conclude that

$$D_p = (-1)^n p^{(n)} \det \begin{pmatrix} 0 & p(1) - p(0) & \cdots & p(n) - p(n-1) \\ 0 & p(0) - p(-1) & \cdots & p(n-1) - p(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & p(2-n) - p(1-n) & \cdots & p(1) - p(0) \\ 1 & p(1-n) & \cdots & p(0) \end{pmatrix} = p^{(n)} \det(A_{\Delta(p)})$$

We have shown that $D_p = p^{(n)} D_{\Delta(p)}$ for any polynomial $p$ of degree $n$. This proves by an obvious induction that

$$D_p = \prod_{k=0}^{n} (\Delta^k(p))^{(n-k)}.$$

But, clearly $(\Delta^k(p))^{(n-k)} = \Delta^k(p^{(n-k)})$, and using the lemma and the fact that deg $p^{(n-k)} = k$ we obtain $\Delta^k(p^{(n-k)}) = p^{(n)}$, so $(\Delta^k(p))^{(n-k)} = p^{(n)}$. Therefore, $D_p = (p^{(n)})^{n+1}$.

Finally if $p$ is monic then $p^{(n)} = n!$. Hence, for a monic polynomial $p$ of degree $n$ we have $D_p = (n!)^{n+1}$. $\square$

**Proof of the Lemma.** First, note that $\Delta = T - I$ Where $I$ is the identity operator, and $T$ is the translation operator defined by $T(Q)(X) = Q(X + 1)$. So the first equality follows from the binomial formula $\Delta^n = (T-I)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} T^k$.

Now, to prove that $\Delta^n(Q) = Q^{(n)}$ for $Q \in \mathbb{C}_n[X]$, we only need to check this when $Q$ is one of the elements of the basis $\{Q_0, \ldots, Q_n\}$ defined by $Q_0 = 1$ and $Q_k = \frac{1}{k!}X(X+1)\ldots(X+k-1)$ for $k \geq 1$, and this is easy since $\Delta(Q_k) = Q_{k-1}$. 

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Problem 11379. Proposed by Oskar Maria Baksalary, Adam Mickiewicz University, Poznań, Poland, and Götz Trenkler, Technische Universität Dortmund, Dortmund, Germany. Let $A$ be a complex matrix of order $n$ whose square is the zero matrix. Show that $\mathcal{R}(A+A^*) = \mathcal{R}(A) + \mathcal{R}(A^*)$, $\mathcal{R}(\cdot)$ denotes the column space of a matrix argument.

Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

We will identify a complex square matrix $B$ of order $n$ with the linear operator $L_B : x \mapsto Bx$ on the standard inner product space $\mathbb{C}^n$, so that $\mathcal{R}(B) = \text{im}(L_B)$.

Let us recall some well-known results from linear algebra. For an operator $\Phi : \mathbb{C}^n \rightarrow \mathbb{C}^n$ we have $(\text{im}(\Phi))^\perp = \ker(\Phi^*)$, and for subspaces $V$ and $W$ in $\mathbb{C}^n$ we have $(V+W)^\perp = V^\perp \cap W^\perp$, and finally $V = W$ if, and only if, $V^\perp = W^\perp$.

Now, we have,

$$(\mathcal{R}(A + A^*))^\perp = (\text{im}(L_{A+A^*}))^\perp = \ker((L_{A+A^*})^*) = \ker(L_{(A+A^*)^*}) = \ker(L_{A^*+A})$$

and

$$(\mathcal{R}(A) + \mathcal{R}(A^*))^\perp = (\mathcal{R}(A))^\perp \cap (\mathcal{R}(A^*))^\perp = \ker(L_{A^*}) \cap \ker(L_A)$$

so, we have to show that, if $A^2 = 0$ then

$$\ker(L_{A^*+A^*}) = \ker(L_A) \cap \ker(L_{A^*})$$

Clearly, we have $\ker(L_A) \cap \ker(L_{A^*}) \subset \ker(L_{A^*+A^*})$. Conversely, let us consider $x \in \ker(L_{A^*+A^*})$, that is $Ax = -A^*x$. It follows that $A^*Ax = -A^*A^*x = -(A^2)x = 0$, and consequently, $\|Ax\|^2 = x^*A^*Ax = 0$ so that $Ax = 0$ and $A^*x = -Ax = 0$, hence, $x \in \ker(L_A)$ and $x \in \ker(L_{A^*})$. This achieves the proof. \qed
Problem 11381. Proposed by Jesús Guillera, Zaragoza, Spain, and Jonathan Sondow, New York, NY.
Show that if \( x \) is a positive real number, then
\[
e^x = \prod_{n=1}^{\infty} \left( \prod_{k=0}^{n} (kx + 1)^{(n-k+1)}(x)^{-k} \right)^{1/n}.
\]

Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria).
For \( n \geq 1 \), let \( F_n(X) \) denote the rational fraction given by
\[
F_n(X) = \frac{(n-1)!X^{n-1}}{(1+X)(1+2X)\cdots(1+nX)}
\]
We know that there are real coefficients \( (a_1, a_2, \ldots, a_n) \) such that \( F_n(X) = \sum_{k=1}^{n} \frac{a_k}{1+kX} \), and we can calculate \( a_j \), for \( 1 \leq j \leq n \) as follows,
\[
a_j = \lim_{x \to -1/j} (1+jx)F_n(x) = (-1)^{j+1} \frac{j}{n} \binom{n}{j},
\]
therefore,
\[
F_n(X) = \frac{1}{n} \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \frac{k}{1+kX}.
\]
We conclude that
\[
\int_{0}^{x} F_n(t) \, dt = \frac{1}{n} \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \ln(1+kx) = \frac{1}{n} \ln \left( \prod_{k=0}^{n} (kx + 1)^{(n-k+1)}(x)^{-k} \right)
\]
On the other hand, \( F_1(X) = 1 - \frac{X}{1+X} \) and for \( n > 1 \), we have
\[
F_n(X) = \frac{(n-1)!X^{n-1}}{(1+X)(1+2X)\cdots(1+(n-1)X)} - \frac{nX^n}{(1+X)(1+2X)\cdots(1+nX)}
\]
So, if we define the sequence of functions \( (g_n)_{n \geq 0} \) on \( \mathbb{R}_+^* \) by \( g_0(x) = x \) and, for \( n > 0 \),
\[
g_n(x) = \int_{0}^{x} \frac{n!t^n}{(1+t)(1+2t)\cdots(1+nt)} \, dt
\]
then
\[
\forall n \geq 1, \quad \int_{0}^{x} F_n(t) \, dt = g_{n-1}(x) - g_n(x).
\]
It follows from (*) that,
\[
x - g_m(x) = \sum_{n=1}^{m} \int_{0}^{x} F_n(t) \, dt = \ln \left( \prod_{n=1}^{m} \left( \prod_{k=0}^{n} (kx + 1)^{(n-k+1)}(x)^{-k} \right)^{1/n} \right).
\]
so, to obtain the desired result, we have to prove that \( \lim_{m \to \infty} g_m(x) = 0 \).
For \( k \geq 1 \) and \( 0 \leq t \leq x \) we have
\[
0 \leq \frac{kt}{1+kt} = 1 - \frac{1}{1+kt} \leq 1 - \frac{1}{k(1+x)} \leq \exp \left( -\frac{1}{k(1+x)} \right),
\]
so that
\[
0 \leq g_m(x) \leq x \exp \left( -\frac{H_m}{1+x} \right)
\]
with \( H_m = \sum_{k=1}^{m} 1/k \). Finally, since \( \lim_{m \to \infty} H_m = +\infty \), we conclude that \( \lim_{m \to \infty} g_m(x) = 0 \), and this leads to the desired conclusion.
Problem 11382. Proposed by Roberto Tauraso, Università di Roma “Tor Vergata”, Roma, Italy. For \( k \geq 1 \), let \( H_k \) be the \( k \)th harmonic number, \( H_k = \sum_{j=1}^{k} 1/j \). Show that if \( p \) is a prime and \( p > 5 \), then

\[
\sum_{k=1}^{p-1} \frac{H_k^2}{k} \equiv \sum_{k=1}^{p-1} \frac{H_k}{k^2} \pmod{p^2}.
\]

(Two rationals are congruent \( \pmod{d} \) if their difference can be expressed as a reduced fraction of the form \( da/b \) with \( b \) relatively prime to \( a \) and \( d \).)

Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria). For \( 1 \leq k < p \), we have

\[
H_k^3 - H_{k-1}^3 = \frac{1}{k} = H_k - \left( H_k - \frac{1}{k} \right)^3 = \frac{1}{k} = 3 \left( \frac{H_k^2}{k} - \frac{H_k}{k^2} \right)
\]

with the convention \( H_0 = 0 \). Therefore, by summing these equalities we obtain,

\[
3 \left( \sum_{k=1}^{p-1} \frac{H_k^2}{k} - \sum_{k=1}^{p-1} \frac{H_k}{k^2} \right) = H_p^3 - 1 \sum_{k=1}^{p-1} \frac{1}{k^3}.
\]

Now, the statement of the problem follows from the following lemma, and the fact that \( \gcd(p, 3) = 1 \) for any prime \( p > 5 \).

Lemma.

1º. For a prime \( p > 2 \) we have \( H_{p-1} \equiv 0 \pmod{p} \).  
2º. For a prime \( p > 5 \) we have \( \sum_{k=1}^{p-1} 1/k^3 \equiv 0 \pmod{p^2} \).

Proof.

1º. This is a weaker version of Wolstenholme’s theorem, which can be proved as follows

\[
2H_{p-1} = \sum_{k=1}^{p-1} \left( \frac{1}{k} + \frac{1}{p-k} \right) = p \sum_{k=1}^{p-1} \frac{1}{k(p-k)}
\]

therefore, \( 2H_{p-1} \equiv 0 \pmod{p} \) since the common denominator of the fractions in the sum \( \sum_{k=1}^{p-1} 1/(k(p-k)) \) is coprime with \( p \). This proves 1º because \( \gcd(p, 2) = 1 \).

2º. In fact,

\[
2 \sum_{k=1}^{p-1} \frac{1}{k^3} = \sum_{k=1}^{p-1} \left( \frac{1}{k^3} + \frac{1}{(p-k)^3} \right) = \sum_{k=1}^{p-1} \frac{p^3 - 3p^2k + 3pk^2}{k^3(p-k)^3} = p^3 \sum_{k=1}^{p-1} \frac{1}{k^3(p-k)^3} - 3p^2 \sum_{k=1}^{p-1} \frac{1}{k^2(p-k)^3} + 3p \sum_{k=1}^{p-1} \frac{1}{k(p-k)^3}.
\]

Since \( p \) is coprime with the common denominator of the fractions in each of the sums \( \sum_{k=1}^{p-1} 1/k^3(p-k)^3 \) and \( \sum_{k=1}^{p-1} 1/(k(p-k))^3 \) we conclude from the above equality that

\[
2 \sum_{k=1}^{p-1} \frac{1}{k^3} \equiv 3p \sum_{k=1}^{p-1} \frac{1}{k(p-k)^3} \pmod{p^2}.
\]

Now, it is clear that 2º would follow, if we prove that for a prime \( p > 5 \) we have \( \sum_{k=1}^{p-1} 1/(k(p-k)^3) \equiv 0 \pmod{p} \).
Let $B$ denote $((p-1)!)^4$ which is a common denominator to the fractions in this sum, and let the integer $A$ be defined by

$$A = B \sum_{k=1}^{p-1} \frac{1}{k(p-k)^3}$$

then we have to show that $A \equiv 0 \mod p$ for any prime $p > 5$.

For $1 \leq k < p$, let $k^{-1}$ denote the inverse of $k$ modulo $p$, that is the unique element from $\{1, 2, \ldots, p-1\}$ satisfying $kk^{-1} \equiv 1 \mod p$, then

$$\forall k \in \{1, \ldots, p-1\}, \quad \frac{B}{k(p-k)^3} \equiv -(k^{-1})^4 \mod p.$$  

(By Wilson’s theorem $(p-1)! \equiv -1 \mod p$ so that $B \equiv 1 \mod p$.) It follows that $A \equiv -\sum_{k=1}^{p-1} k^{-4} \mod p$.

In the finite field $\mathbb{F}_p$ the polynomial $X^4 - 1$ has at most four zeros, so if $p > 5$ then we can find an element $a$ in $\{1, \ldots, p-1\}$ such that $a^4 - 1 \not\equiv 0 \mod p$. Now, the mapping $k \mapsto a^{-1}k \mod p$ is a permutation of the set $\{1, \ldots, p-1\}$, (the inverse being $k \mapsto ak \mod p$.) So, $\sum_{k=1}^{p-1} k^{-4} \equiv a^4 \sum_{k=1}^{p-1} k^{-4} \mod p$ and consequently $(a^4 - 1) \sum_{k=1}^{p-1} k^{-4} \equiv 0 \mod p$, but $p$ and $a^4 - 1$ are coprime, by our choice of $a$, so we must have $\sum_{k=1}^{p-1} k^{-4} \equiv 0 \mod p$, that is $A \equiv 0 \mod p$. This ends the proof of the lemma, and achieves the solution of the problem. 

\[\square\]
Problem 11383. Proposed by Micheal Nublom, RMIT University, Australia. Show that

\[
\sum_{n=1}^{\infty} \cos^{-1} \left( \frac{1 + \sqrt{n^2 + 2n\sqrt{n^2 + 4n + 3}}}{(n+1)(n+2)} \right) = \frac{\pi}{6}.
\]

Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

Contrary to the original statement of the problem, the right answer is \(\pi/6\) and not \(\pi/3\). To see this, let \(\theta_n = \cos^{-1}(1/(n+1)) \in (0, \pi/2)\), then

\[
\cos(\theta_{n+1} - \theta_n) = \cos \theta_{n+1} \cos \theta_n + \sin \theta_{n+1} \sin \theta_n = \frac{1}{n+1} \cdot \frac{1}{n+2} + \sqrt{1 - \frac{1}{(n+1)^2}} \cdot \sqrt{1 - \frac{1}{(n+2)^2}}.
\]

But \(0 < \theta_{n+1} - \theta_n < \pi/2\), therefore,

\[
\cos^{-1} \left( \frac{1 + \sqrt{n^2 + 2n\sqrt{n^2 + 4n + 3}}}{(n+1)(n+2)} \right) = \theta_{n+1} - \theta_n,
\]

and consequently,

\[
\sum_{n=1}^{m} \cos^{-1} \left( \frac{1 + \sqrt{n^2 + 2n\sqrt{n^2 + 4n + 3}}}{(n+1)(n+2)} \right) = \theta_m - \theta_1 = \cos^{-1} \left( \frac{1}{m+1} \right) - \frac{\pi}{3}.
\]

so, letting \(m\) tend to \(+\infty\), we find that

\[
\sum_{n=1}^{\infty} \cos^{-1} \left( \frac{1 + \sqrt{n^2 + 2n\sqrt{n^2 + 4n + 3}}}{(n+1)(n+2)} \right) = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}. \quad \Box
\]
Problem 11384. Proposed by MOUBINOL OMARJEE, Lycée Jean-Lurçat, Paris, France. Let $p_n$ denote the $n^{th}$ prime. Show that

$$
\sum_{n=1}^{\infty} \frac{(-1)^{\lfloor \sqrt{n} \rfloor}}{p_n}
$$

converges.

Solution, by OMRAK KOUBA (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

More generally, We will prove that if $(a_n)_{n \geq 1}$ is a nonincreasing sequence of nonnegative real numbers, such that $\sum a_n/\sqrt{n}$ converges, then $\sum(-1)^{\lfloor \sqrt{n} \rfloor}a_n$ also converges.

The conclusion of the problem follows since $(1/p_n)_{n \geq 1}$ is clearly decreasing and satisfy $\frac{1}{p_n} \sim \frac{1}{n \ln n}$ by the Prime Number Theorem.

We will use the following lemma.

Lemma 1. Let us define $A_n = \sum_{j=0}^{n}(-1)^{\lfloor \sqrt{j} \rfloor}$, for $n \geq 0$. Then, there exists a constant $\lambda$ such that, for all $n \geq 1$ we have $|A_n| \leq \lambda \sqrt{n}$.

Proof. The key idea is to note that $(-1)^{\lfloor \sqrt{j} \rfloor} = 1$ for $(2k)^2 \leq j < (2k + 1)^2$ and $(-1)^{\lfloor \sqrt{j} \rfloor} = -1$ for $(2k + 1)^2 \leq j < (2k + 2)^2$, so that

$$
\sum_{j=(2k)^2}^{(2k+2)^2-1} (-1)^{\lfloor \sqrt{j} \rfloor} = ((2k + 1)^2 - 4k^2) - (4(k + 1)^2 - (2k + 1)^2) = -2.
$$

Now, let $n$ be a positive integer and define $\ell = \lfloor \sqrt{n}/2 \rfloor$ then $4\ell^2 \leq n < 4(\ell + 1)^2$ so that $n = 4\ell^2 + r$ with $0 \leq r \leq 8\ell + 3$. Clearly,

$$
|A_n - A_{4\ell^2-1}| \leq \sum_{j=4\ell^2}^{n} 1 = n - (4\ell^2 - 1) = r + 1 \leq 8\ell + 4,
$$

and

$$
A_{4\ell^2-1} = \sum_{k=0}^{\ell - 1} \left( \sum_{j=(2k)^2}^{(2k+2)^2-1} (-1)^{\lfloor \sqrt{j} \rfloor} \right) = -2\ell.
$$

It follows that, for $n \geq 1$, we have

$$
|A_n| \leq 2\ell + 8\ell + 4 \leq 10\ell + 4 \leq 5\sqrt{n} + 4 \leq 9\sqrt{n}
$$

this ends the proof of lemma 1 with $\lambda = 9$, (which is not the best possible.)

Lemma 2. Let $(a_n)_{n \geq 1}$ be a nonincreasing sequence of nonnegative real numbers, such that $\sum a_n/\sqrt{n}$ converges, then $\sum(-1)^{\lfloor \sqrt{n} \rfloor}a_n$ also converges.

Proof. We will use Abel transformation as follows, for $m$ and $n$ two integers such that $m > n > 1$ we have,

$$
\sum_{k=n}^{m} a_k(-1)^{\lfloor \sqrt{k} \rfloor} = \sum_{k=n}^{m} a_k(A_k - A_{k-1}) = \sum_{k=n}^{m} a_kA_k - \sum_{k=n-1}^{m-1} a_{k+1}A_k
$$

$$
= a_mA_{m+1} - a_nA_{n-1} + \sum_{k=n}^{m} (a_k - a_{k+1})A_k.
$$
It follows that,
\[
\sum_{k=n}^{m} a_k(-1)^{\lfloor \sqrt{k} \rfloor} \leq \lambda \left( \sqrt{m+1} a_m + \sqrt{n a_n} + \sum_{k=n}^{m} (a_k - a_{k+1}) \sqrt{k} \right)
\]
\[
\leq \lambda \left( \sqrt{m+1} a_m + \sqrt{n a_n} + \sum_{k=n}^{m} a_k \sqrt{k} - \sum_{k=n+1}^{m+1} a_k \sqrt{k-1} \right)
\]
\[
\leq \lambda \left( \sqrt{n a_n} + a_n \sqrt{n-1} + \sum_{k=n}^{m} a_k (\sqrt{k} - \sqrt{k-1}) \right)
\]
\[
\leq \lambda \left( 2 \sqrt{n a_n} + \sum_{k=n}^{m} \frac{a_k}{\sqrt{k}} \right)
\]
So we have shown that, for \( m > n > 1 \) we have
\[
\left| \sum_{k=n}^{m} a_k(-1)^{\lfloor \sqrt{k} \rfloor} \right| \leq \lambda \left( 2 \sqrt{n a_n} + \sum_{k=n}^{\infty} \frac{a_k}{\sqrt{k}} \right) \tag{1}
\]
But, it is well-known that if \((b_n)_{n \geq 1}\) is a non-increasing sequence of real numbers such that \(\sum b_n\) converges then \(\lim_{n \to \infty} n b_n = 0\). So the hypotheses on the sequence \((a_n)_{n \geq 1}\) imply (with \(b_n = a_n/\sqrt{n}\)), that
\[
\lim_{n \to \infty} \sqrt{n a_n} = 0 \quad \text{and} \quad \lim_{n \to \infty} \sum_{k=n}^{\infty} \frac{a_k}{\sqrt{k}} = 0.
\]
Therefore, given \(\epsilon > 0\) there is \(n_0\) such that for \(n > n_0\) we have
\[
2 \sqrt{n a_n} + \sum_{k=n}^{\infty} \frac{a_k}{\sqrt{k}} < \frac{\epsilon}{\lambda}.
\]
Now, for \(m > n > n_0\) we obtain using (1) that
\[
\left| \sum_{k=n}^{m} a_k(-1)^{\lfloor \sqrt{k} \rfloor} \right| \leq \epsilon
\]
this proves that the sequence of partial sums of the series \(\sum a_k(-1)^{\lfloor \sqrt{k} \rfloor}\) is a Cauchy sequence, and the convergence of the considered series follows.
Problem 11385. Proposed by Jose Luis Diaz-Barrero, Universidad Politécnica de Cataluña; Barcelona, Spain. Let $\alpha_0$, $\alpha_1$ and $\alpha_2$ be the radian measures of the angles of an acute triangle, and for $i \geq 3$ let $\alpha_i = \alpha_{i-3}$. Show that

$$\sum_{i=0}^{2} \frac{\alpha_i^2}{\alpha_{i+1}\alpha_{i+2}}(3 + 2 \tan^2 \alpha_i)^{1/4} \geq 3\sqrt{3}. \quad (1)$$

Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

By the AM – GM inequality we have

$$\frac{1}{3} \sum_{i=0}^{2} \alpha_i^3(3 + 2 \tan^2 \alpha_i)^{1/4} \geq \alpha_0\alpha_1\alpha_2 \left( \prod_{i=0}^{2} (3 + 2 \tan^2 \alpha_i) \right)^{1/12} \quad (1)$$

(with equality if, and only if, $\alpha_0 = \alpha_1 = \alpha_2$, since $x \mapsto x^3(3 + 2 \tan^2 x)$ is strictly increasing on $[0, \pi/2)$.)

On the other hand, if we consider the function $f$ defined on $[0, \pi/2)$ by

$$f(x) = \frac{1}{4} \ln(3 + 2 \tan^2 x) = \frac{1}{4} \ln \left( 1 + \frac{2}{\cos^2 x} \right),$$

then we easily see that

$$f'(x) = \frac{\sin x}{\cos x(2 + \cos^2 x)} \quad \text{and} \quad f''(x) = \frac{2 + \cos^4 x + 3 \cos^2 x \sin^2 x}{\cos^2 x(2 + \cos^2 x)^2}.$$

We conclude that $f$ is convex. Consequently, since $0 < \alpha_i < \pi/2$ and $\alpha_0 + \alpha_1 + \alpha_2 = \pi$ we have

$$f \left( \frac{\pi}{3} \right) = f \left( \frac{\alpha_0 + \alpha_1 + \alpha_2}{3} \right) \leq \frac{f(\alpha_0) + f(\alpha_1) + f(\alpha_2)}{3}$$

that is,

$$\sqrt{3} \leq \left( \prod_{i=0}^{2} (3 + 2 \tan^2 \alpha_i) \right)^{1/12}. \quad (2)$$

The desired inequality follows from (1) and (2), with equality if and only if the triangle $ABC$ is equilateral. $\square$
Problem 11386. Proposed by Greg Markowsky, Somerville, MA. Consider a triangle $ABC$. Let $O$ be the circumcircle of $ABC$, $r$ the radius of the incircle, and $s$ the semiperimeter. Let $\text{arc}(BC)$ be the arc of $O$ opposite $A$, and define $\text{arc}(CA)$ and $\text{arc}(AB)$ similarly. Let $O_A$ be the circle tangent to $AB$ and $AC$ and internally tangent to $O$ along $\text{arc}(BC)$, and let $R_A$ be its radius. Define $O_B$, $O_C$, $R_B$, and $R_C$ similarly. Show that

$$\frac{1}{aR_A} + \frac{1}{bR_B} + \frac{1}{cR_C} = \frac{s^2}{rabc}.$$ 

Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria). We will use the following lemma.

Lemma. Let $I$ denote the incenter of the incircle $C$ of the triangle $ABC$, and $C_A$ denote the circle with center $A$ and radius $AI$. Then the radical axis $\Delta_A$ of $C$ and $C_A$ is tangent to the incircle $C$.

Proof. let $O$ be the circumcenter, and consider the point $X$ from $C$ defined by $\overrightarrow{TX} = \frac{1}{2}\overrightarrow{OA}$. Then

$$OX^2 - AX^2 = (\overrightarrow{OX} - \overrightarrow{AX}) \cdot (\overrightarrow{OX} + \overrightarrow{AX}) = \overrightarrow{OA} \cdot (\overrightarrow{XI} + \overrightarrow{XI} + 2\overrightarrow{OX})$$

$$= \overrightarrow{OA} \cdot (\overrightarrow{XI} + \overrightarrow{XI} + 2r\overrightarrow{OA}) = 2rR + (\overrightarrow{OI} - \overrightarrow{AI}) \cdot (\overrightarrow{OI} + \overrightarrow{AI})$$

$$= 2rR + OI^2 - AI^2.$$ 

But it is known that $OI^2 = R^2 - 2rR$, hence $OX^2 - AX^2 = R^2 - AI^2$, and we conclude that

$$OX^2 - R^2 = AX^2 - AI^2.$$ 

This means that the power of $X$ with respect to $O$ is the same as its power with respect to $C_A$, and hence $X$ lies on the radical axis $\Delta_A$ of $O$ and $C_A$. So $\Delta_A$ is the line through $X$ perpendicular to $OA$, but $OA \parallel IX$, so $\Delta_A$ is the line through $X$ perpendicular to $IX$, and this is the tangent to $C_A$ at $X$. The proof of the lemma is complete. \hfill \square

Let us now come to the proposed problem. Let $\mathbf{I}$ denote the inversion with respect to the circle $C_A$. Clearly, the extended lines $AB$ and $AC$ are globally invariant under the inversion $\mathbf{I}$ and $\mathbf{I}(\Delta_A) = O$. So, by the lemma, we must have $\mathbf{I}(C) = O_A$.

Let the segment $AI$ intersect $C$ in $N$, and let $M$ be the second point of intersection of $C$ with the line $AI$. Finally, let $\mathbf{M'} = \mathbf{I}(M)$ and $\mathbf{N'} = \mathbf{I}(N)$. Since the segment $NM$ is a diameter in $C$ then its image $N'M'$ by the inversion $\mathbf{I}$ is a diameter in $O_A$.

We have $AN' \cdot AN = AI^2$ and $AM' \cdot AM = AI^2$, so that

$$2R_A = AN' - AM' = AI^2 \left( \frac{1}{AN} - \frac{1}{AM} \right) = AI^2 \frac{2r}{AM \cdot AN}.$$ 

Now, if $Q$ is the point where $AC$ meets $C$ then $AM \cdot AN = AQ^2$ since this is the power of $A$ with respect to $C$. We conclude that

$$\frac{r}{R_A} = \frac{AQ^2}{AI^2} = \cos^2 \left( \frac{A}{2} \right) = \frac{1 + \cos A}{2}.$$ 

1
and consequently,

$$\frac{4rabc}{aR_A} = 2bc + 2bc \cos \hat{A} = b^2 + c^2 - a^2 + 2bc$$

Therefore, by adding similar relations, for $B$ and $C$ we find that

$$4rabc \left( \frac{1}{aR_A} + \frac{1}{bR_B} + \frac{1}{cR_C} \right) = (a + b + c)^2 = 4s^2,$$

and the desired result follows immediately.

**Remark.** The relation $OI^2 = R(R - 2r)$ can be proved as follows: Noting that $I$ is the barycenter of the points $(A; a)$, $(B; b)$ and $(C; c)$, we have

$$OI = \frac{1}{2s} \left( a\overrightarrow{OA} + b\overrightarrow{OB} + c\overrightarrow{OC} \right)$$

therefore,

$$OI^2 = \frac{R^2}{4s^2} \left( a^2 + b^2 + c^2 + 2ab \cos 2\hat{C} + 2bc \cos 2\hat{A} + 2ca \cos 2\hat{B} \right)$$

$$= \frac{R^2}{4s^2} \left( (a + b + c)^2 - 4ab \sin^2 \hat{C} - 4bc \sin^2 \hat{A} - 4ca \sin^2 \hat{B} \right)$$

$$= \frac{R^2}{4s^2} \left( 4s^2 - 8A \sin \hat{C} \sin \hat{A} \sin \hat{B} \right)$$

$$= \frac{R^2}{4s^2} \left( 4s^2 - \frac{8sr}{2R} (a + b + c) \right) = R^2 - 2rR.$$
Problem 11387. Proposed by Oskar Maria Baksalary, Adam Mickiewicz University, Poznań, Poland, and Götz Trenkler, Technische Universität Dortmund, Dortmund, Germany. Let $C_{n,n}$ denote the set of $n \times n$ complex matrices. Determine the shortest interval $[a, b]$ such that if $P$ and $Q$ in $C_{n,n}$ are nonzero orthogonal projectors, that is, Hermitian idempotent matrices, then all eigenvalues of $PQ + QP$ belong to $[a, b]$.

Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

The answer is $[-1/4, 2]$ for $n \geq 2$, and it is trivially reduced to the singleton $\{2\}$ for $n = 1$. In the sequel we will only consider the case $n > 1$. Our solution is based upon the following lemma.

**Lemma.** Let $(E, \langle \cdot, \cdot \rangle)$ be a complex inner product space, with $\dim E > 1$, and let $S$ denote the unit sphere in $E$. For $(x, y, z) \in S^3$ we define

$$F(x, y, z) = 2 \Re(\langle x, y \rangle \langle y, z \rangle \langle z, x \rangle) = \langle x, y \rangle \langle y, z \rangle \langle z, x \rangle + \langle z, y \rangle \langle x, z \rangle \langle y, x \rangle,$$

then,

$$\min \{ F(x, y, z) : (x, y, z) \in S^3 \} = -\frac{1}{4}.$$

**Proof.**

Consider $(x, y, z)$ in $S^3$. Since the gram-matrix

$$G(x, y, z) = \begin{bmatrix}
1 & \langle x, y \rangle & \langle x, z \rangle \\
\langle y, x \rangle & 1 & \langle x, z \rangle \\
\langle z, x \rangle & \langle z, y \rangle & 1
\end{bmatrix},$$

is a Hermitian positive matrix, we conclude that $\det G(x, y, z) \geq 0$, that is

$$1 + F(x, y, z) - |\langle x, y \rangle|^2 - |\langle y, z \rangle|^2 - |\langle z, x \rangle|^2 \geq 0$$

or, equivalently,

$$2t^3 - |\langle x, y \rangle|^2 - |\langle y, z \rangle|^2 - |\langle z, x \rangle|^2 + 1 \geq 0 \quad (1)$$

with $t = \sqrt[3]{F(x, y, z)}/2$. On the other hand, by the AM-GM inequality we have

$$|\langle x, y \rangle|^2 + |\langle y, z \rangle|^2 + |\langle z, x \rangle|^2 \geq 3 \sqrt[3]{|\langle x, y \rangle|^2 |\langle y, z \rangle|^2 |\langle z, x \rangle|^2} = 3 \sqrt[3]{\frac{F(x, y, z)}{2}}^2 \geq 3t^2. \quad (2)$$

From (1) and (2) we conclude that $2t^3 - 3t^2 + 1 \geq 0$ which is equivalent to $(t-1)^2(2t+1) \geq 0$. This proves that $2t + 1 \geq 0$, or equivalently, $F(x, y, z) \geq -1/4$. So we have proved that $\inf \{ F(x, y, z) : (x, y, z) \in S^3 \} \geq -\frac{1}{4}$.

To see that this lower bound is in fact a minimum, we note that, since $\dim E > 1$, there are two orthonormal vectors $u$ and $v$ in $E$, and then one checks easily that,

$$F \left( u, -\frac{1}{2}u + \frac{\sqrt{3}}{2}v, -\frac{1}{2}u - \frac{\sqrt{3}}{2}v \right) = -\frac{1}{4}.$$

This achieves the proof of the lemma.
Let us now consider two nonzero Hermitian projectors \( P \) and \( Q \) from \( C_{n,n} \) with \( n > 1 \), and let \( U = PQ + QP \), then \( U \) is Hermitian since
\[
U^* = Q^* P^* + P^* Q^* = QP + PQ = U
\]
It follows that the eigenvalues of \( U \) are real, and that if \( \lambda(P, Q) \) and \( \Lambda(P, Q) \) denote respectively the smallest and the largest eigenvalues of \( U \) then
\[
\lambda(P, Q) = \min_{\|x\|=1} \langle Ux, x \rangle \quad \text{and} \quad \Lambda(P, Q) = \max_{\|x\|=1} \langle Ux, x \rangle.
\]
Where \( \langle , \rangle \) is the standard inner product in \( C_{n,1} \equiv C^n \). But,
\[
\langle Ux, x \rangle = \langle PQx, x \rangle + \langle QPx, x \rangle = \langle Qx, Px \rangle + \langle Px, Qx \rangle = 2 \Re(\langle Px, Qx \rangle)
\]
So,
\[
\lambda(P, Q) = 2 \min_{\|x\|=1} \Re(\langle Px, Qx \rangle) \quad \text{and} \quad \Lambda(P, Q) = 2 \max_{\|x\|=1} \Re(\langle Px, Qx \rangle). \quad (3)
\]
Consider two Hermitian projectors \( P \) and \( Q \) from \( C_{n,n} \). For \( x \in C_{n,1} \) such that \( \|x\| = 1 \) we have, by the Cauchy-Schwarz inequality, that
\[
\Re(\langle Px, Qx \rangle) \leq \|Px\| \|Qx\| \leq \|x\|^2 = 1.
\]
It follows that \( \Lambda(P, Q) \leq 2 \). Moreover, if \( I_n \) denotes the identity matrix then \( \Lambda(I_n, I_n) = 2 \). We conclude that \( 2 \) is the maximum value assumed by \( \Lambda(P, Q) \) when the matrices \( P \) and \( Q \) run over the set of Hermitian projectors in \( C_{n,n} \).

On the other hand, Let \( P \) and \( Q \) be two Hermitian projectors from \( C_{n,n} \), and let \( x \) be a vector from \( C_{n,1} \) such that \( \|x\| = 1 \).
If \( Px \neq 0 \) and \( Qx \neq 0 \), we set
\[
y = \frac{1}{\|Px\|} Px \quad \text{and} \quad z = \frac{1}{\|Qx\|} Qx
\]
then \( \langle x, y \rangle = Px, \langle x, z \rangle = Qx, \) and
\[
\langle Px, Qx \rangle = \langle (x, y) \ 0, (x, z) \rangle = \langle x, y \rangle \langle y, z \rangle \langle z, x \rangle,
\]
so, using the notation of the lemma, we have \( 2 \Re(\langle Px, Qx \rangle) = F(x, y, z) \), and consequently, \( 2 \Re(\langle Px, Qx \rangle) \geq -1/4 \). Clearly, this conclusion remains valid if \( Px = 0 \) or \( Qx = 0 \), so we have proved that \( \lambda(P, Q) \geq -1/4 \). Moreover, as in the proof of the lemma, if \( u \) and \( v \) denote two orthonormal vectors in \( C_{n,1} \) and we define
\[
y = -\frac{1}{2} u + \frac{\sqrt{3}}{2} v, \ z = -\frac{1}{2} u - \frac{\sqrt{3}}{2} v,
\]
and we consider the Hermitian projectors \( P_0 \) and \( Q_0 \) defined, for \( x \in C_{n,1} \), by \( P_0 x = (x, y) \ y, \ Q_0 x = (x, z) \ z \), then we have immediately
\[
-\frac{1}{4} \leq \lambda(P_0, Q_0) \leq 2 \Re(\langle P_0 u, Q_0 u \rangle) = -\frac{1}{4},
\]
that is \( \lambda(P_0, Q_0) = -1/4 \). We conclude that \( -1/4 \) is the minimum value assumed by \( \lambda(P, Q) \) when the matrices \( P \) and \( Q \) run over the set of Hermitian projectors in \( C_{n,n} \). This achieves the proof.
Problem 11392. Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Let the consecutive vertices of a regular convex \( n \)-gon \( P \) be denoted \( A_0, \ldots, A_{n-1} \), in order, and let \( A_n = A_0 \). Let \( M \) be a point such that for \( 0 \leq k < n \) the perpendicular projections of \( M \) onto each line \( A_kA_{k+1} \) lie interior to the segment \((A_k, A_{k+1})\). Let \( B_k \) be the projection of \( M \) onto \( A_kA_{k+1} \). Show that
\[
\sum_{k=0}^{n-1} \text{Area}(\triangle(MA_kB_k)) = \frac{1}{2} \text{Area}(P).
\]

Solution, by the proposer.

For \( 0 \leq k < n \), let the complex number \( z_k \) represent the point \( A_k \), and let \( z \) represent the point \( M \). Without loss of generality, we may assume that \( 0 \) represents the centroid of the polygon, and that the length of the side of \( P \) is equal to 1. Then, clearly we have \( z_k = \omega^k z_0 \) with \( \omega = \exp\left(\frac{2\pi i}{n}\right) \), and \( |z_0| |\omega - 1| = 1 \).

Now, the number \( \Re\left((z - z_k)(z_{k+1} - z_k)\right) \) represents the vector \( \overrightarrow{A_kB_k} \), and \( z - z_k \) represents the vector \( \overrightarrow{A_kM} \), so the area of \( \triangle(MA_kB_k) \) is given by
\[
\text{Area}(\triangle(MA_kB_k)) = \frac{1}{4} \Re\left((z - z_k)(z_{k+1} - z_k)\right) \Im\left((z - z_k)(z_{k+1} - z_k)\right)
= \frac{1}{4} \Re\left((z - z_k)(z_{k+1} - z_k)\right) \Im\left((z - z_k)(z_{k+1} - z_k)\right)
= \frac{1}{4} \Re\left((z - z_k)(z_{k+1} - z_k)\right) \Im\left((z - z_k)(z_{k+1} - z_k)\right)
= \frac{1}{4} \Re\left((z - z_k)^2 \omega^{2k}(\omega - 1)^2 z_0^2\right).
\]

But,
\[
\sum_{k=0}^{n-1} (z - z_k)^2 \omega^{2k} = z^2 \sum_{k=0}^{n-1} \omega^{2k} - 2z z_0 \sum_{k=0}^{n-1} \omega^k + n z_0^2
= z^2 \frac{\omega^{2n} - 1}{\omega^2 - 1} - 2z z_0 \frac{\omega^n - 1}{\omega - 1} + n z_0^2 = nz_0^2,
\]
so that
\[
\sum_{k=0}^{n-1} \text{Area}(\triangle(MA_kB_k)) = \frac{1}{4} \Re\left((z - z_k)^2 \omega^{2k}(\omega - 1)^2 z_0^2\right)
= \frac{n}{4} \Re\left((z - z_k)^2 \omega^{2k}(\omega - 1)^2 z_0^2\right).
\]

In particular, the right hand side of this equality is independent of \( z \), (in other words, it is independent of the position of \( M \).) And clearly, if \( M \) is the centroid \( O \) of \( P \), then the left hand side of the preceding equality is equal to \( \frac{1}{2} \text{Area}(P) \). This ends the proof.

My Proposal was published in AMM November 2008.
Problem 11393. Proposed by Cosmin Pohoata (student), National College “Tudor Vianu”, Bucharest, Romania. In triangle $ABC$, let $M$ and $Q$ be points on segment $AB$, and similarly let $N$ and $R$ be points on segment $AC$, and $P$ and $S$ be points on segment $BC$. Let $d_1$ be the line through $M$ and $N$, $d_2$ the line through $P$ and $Q$, and $d_3$ through $R$ and $S$. Let $\rho(X,Y,Z)$ denote the ratio of the length of $XZ$ to that of $XY$. Let $m = \rho(M,A,B)$, $n = \rho(N,A,C)$, $p = \rho(P,B,C)$, $q = \rho(Q,B,A)$, $r = \rho(R,C,A)$, and $s = \rho(S,C,B)$. Prove that the lines $(d_1, d_2, d_3)$ are concurrent if and only if $mpr + nqs + mq + nr + ps = 1$.

Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

We consider $A$ as origin of the affine plane of the triangle $ABC$, and the vectors $\overrightarrow{u} = \overrightarrow{AB}$ and $\overrightarrow{v} = \overrightarrow{AC}$ as basis for a system of affine coordinates in this plane.

Using the definition of the ratios $m,n,p,q,r,s$ one concludes easily that the coordinates of the points $M, N, P, Q, R, S$ with respect to the system $(A; \overrightarrow{u}, \overrightarrow{v})$ are as follows:

$$M \left( \frac{1}{1+m}, 0 \right), \; N \left( 0, \frac{1}{1+n} \right), \; P \left( \frac{p}{1+p}, \frac{1}{1+p} \right), \; Q \left( \frac{q}{1+q}, 0 \right), \; R \left( 0, \frac{r}{1+r} \right), \; S \left( \frac{1}{1+s}, \frac{s}{1+s} \right).$$

We deduce easily the equations of the lines $(d_1, d_2, d_3)$ as follows:

$$d_1 : (1+m)x + (1+n)y = 1$$
$$d_2 : (1+q)x + (q-p)y = q$$
$$d_3 : (r-s)x + (1+r)y = r$$

Now, the lines $(d_1, d_2, d_3)$ are concurrent if and only if

$$\det \begin{bmatrix}
1 + m & 1 + n & 1 \\
1 + q & q - p & q \\
r - s & 1 + r & r
\end{bmatrix} = 0.$$

This equivalent to

$$\det \begin{bmatrix}
m & n & 1 \\
1 & -p & q \\
-s & 1 & r
\end{bmatrix} = 0,$$

that is

$$mpr + nqs + mq + nr + ps = 1.$$
Problem 11395. Proposed by M. FARROKHI, D. G. University of Tsukuba, Tsukuba, Japan. Prove that if $H$ is a finite subgroup of the group $G$ of all continuous bijections of $[0,1]$ to itself, then the order of $H$ is 1 or 2.

Solution, by OMRAK KOUBA (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

First, note that the elements of $G$ are strictly monotone functions, since they are continuous, and one to one (i.e. injective) functions on the interval $[0, 1]$.

So, let us consider in $G$ a subgroup $H$ of finite order $m$.

(i) If $I$ denotes the identity mapping : $x \mapsto x$, then $I$ is the only increasing function in $H$. To see this, we consider an increasing function $f$ from $H$, and an arbitrary element $x \in [0, 1]$. If $f(x) \neq x$ then the fact that $f$ is increasing, implies that the sequence $(x_n)_{n \geq 0}$ defined by $x_0 = x$ and $x_{n+1} = f(x_n)$, is strictly monotonous, (increasing if $f(x) > x$ or decreasing if $f(x) < x$,) and this leads to a contradiction since $x_m = f^m(x_0) = x_0$. It follows that $f(x) = x$, that is $f = I$.

(ii) If $H \neq \{I\}$ then $m = 2$. In fact, if $H \neq \{I\}$ there is a decreasing function $h$ in $H$. Now, let $g$ be any element in $H \setminus \{I\}$, then $g \circ h^{-1}$ is an increasing function in $H$, and by (i), $g \circ h^{-1} = I$ we conclude that $g = h$. It follows that $H = \{I, h\}$ and consequently $m = 2$. \qed
Problem 11398. Proposed by Stanley Huang, Jiangzhen Middle School, Huaining, China. Suppose that acute triangle \(ABC\) has its middle-sized angle at \(A\). Suppose further that the incenter \(I\) is equidistant from the circumcenter \(O\) and the orthocenter \(H\). Show that angle \(A\) has measure 60 degrees and that the circumradius of \(IBC\) is the same as that of \(ABC\).

Solution, by Omran Kouba (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

(We do not suppose that \(ABC\) is acute). Let \(a, b, c, R\) and \(s\) denote the lengths of the sides \(BC, CA, AB\), the circumradius of \(ABC\) and \(s\) the semi-perimeter, respectively. We know that,

\[
\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} \quad \text{and} \quad 2s\overrightarrow{OI} = a\overrightarrow{OA} + b\overrightarrow{OB} + c\overrightarrow{OC},
\]

hence,

\[
2s\overrightarrow{OH} = (a+b+c)\overrightarrow{OA} + (a+c)\overrightarrow{OB} + (a+b)\overrightarrow{OC}.
\]

It follows that

\[
4s^2\overrightarrow{OI}^2 = R^2(a^2 + b^2 + c^2 + 2ab\cos 2\hat{C} + 2ac\cos 2\hat{B} + 2bc\cos 2\hat{A})
\]

\[
4s^2\overrightarrow{HI}^2 = R^2((a+b)^2 + (b+c)^2 + (c+a)^2 + 2(b+c)(a+b)\cos 2\hat{A} + 2(a+c)(a+b)\cos 2\hat{B})
\]

Since \(OA = OB = OC = R\) and \((\overrightarrow{OA}, \overrightarrow{OB}) = 2\hat{C}, (\overrightarrow{OB}, \overrightarrow{OC}) = 2\hat{A}\) and \((\overrightarrow{OC}, \overrightarrow{OA}) = 2\hat{B}\),

Now, doing some algebra shows that the equality \(HI = OI\) is equivalent to

\[
a(1 + 2\cos 2\hat{A}) + b(1 + 2\cos 2\hat{B}) + c(1 + 2\cos 2\hat{C}) = 0,
\]

but \(\sin \theta (1 + 2\cos 2\theta) = \sin 3\theta\), so, using the law of sines, we conclude that

\[
\sin 3\hat{A} + \sin 3\hat{B} + \sin 3\hat{C} = 0. \tag{1}
\]

On the other hand,

\[
\sin 3\hat{A} + \sin 3\hat{B} + \sin 3\hat{C} = 2 \sin \frac{3}{2}(\hat{A} + \hat{B}) \cdot \cos \frac{3}{2}(\hat{A} - \hat{B}) + 2 \sin \frac{3}{2}\hat{C} \cdot \cos \frac{3}{2}\hat{C}
\]

\[
= 2 \sin \frac{3}{2}(\pi - \hat{C}) \cdot \cos \frac{3}{2}(\hat{A} - \hat{B}) + 2 \sin \frac{3}{2}(\pi - \hat{A} - \hat{B}) \cdot \cos \frac{3}{2}\hat{C}
\]

\[
= 2 \cos \frac{3}{2}\hat{C} \cdot \cos \frac{3}{2}(\hat{A} - \hat{B}) - 2 \cos \frac{3}{2}(\hat{A} + \hat{B}) \cdot \cos \frac{3}{2}\hat{C}
\]

\[
= -4 \cos \frac{3}{2}\hat{C} \cdot \cos \frac{3}{2}\hat{B} \cdot \cos \frac{3}{2}\hat{A}.
\]

Therefore (1) implies that one of the angles \(\hat{A}, \hat{B}, \hat{C}\) is equal to \(\frac{\pi}{3}\). If the triangle \(ABC\) is equilateral then \(\hat{A} = \frac{\pi}{3}\), and if it is not equilateral then

\[
\min(\hat{A}, \hat{B}, \hat{C}) < \frac{\pi}{3} < \max(\hat{A}, \hat{B}, \hat{C})
\]

and consequently \(\hat{A} = \frac{\pi}{3}\). The first assertion is proved.

Now, since \(I\) is the point of intersection of angle bisectors then \(\overrightarrow{BIC} = \pi - \frac{1}{2}(\hat{B} + \hat{C}) = \frac{2\pi}{3}\), and by the law of sines the circumradius of \(IBC\) is equal to \(\frac{BC}{2\sin BIC} = \frac{BC}{2\sin \frac{2\pi}{3}} = R\). The proof is complete. \(\square\)
Problem 11400. Proposed by Paul Bracken, University of Texas-Pan, Edinburg, TX. Let $\zeta$ be the Riemann zeta function. Evaluate

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(n+1)}$$

in closed form.

**Solution, by Omran Kouba** (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

The answer is $\ln(2\pi) - \frac{1}{4}$. Our starting point is the following lemma, which is well-known in a more general form. We will present a proof of the lemma in order to make our solution self-contained.

**Lemma.** For $x \in (0, 1)$ we have

$$\frac{1}{x} - \pi \cot(\pi x) = \sum_{n=1}^{\infty} \frac{2x}{n^2 - x^2}. \quad (1)$$

**Proof.** Consider $x$ in $(0, 1)$, and let $f$ be the $2\pi$-periodic function, defined on the interval $[-\pi, \pi]$ by $f(t) = \cos(xt)$. The function $f$ is continuous with piecewise continuous derivative, so it must coincide with its Fourier series. The Fourier coefficients $(C_n(f))_{n \in \mathbb{Z}}$ of $f$ are easily determined :

$$C_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(xt)e^{-int}dt = \frac{x \sin(\pi x)}{\pi} \cdot \left( \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2(-1)^n \cos(nt)}{n^2 - x^2} \right),$$

so that, for any real $t$ we have

$$\cos(xt) = \sum_{n \in \mathbb{Z}} C_n(f)e^{int} = \frac{\sin(\pi x)}{\pi} \left( \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2(-1)^n \cos(nt)}{n^2 - x^2} \right),$$

and (1) follows by choosing $t = \pi$. \hfill \Box

Now, consider $x$ in $(0, 1)$. Since the terms of the double series $\sum_{n,k \geq 1} \frac{2n}{k^2 - x^2}$ are positive, we have

$$\sum_{k=1}^{\infty} \frac{x^2/k^2}{1 - x^2/k^2} = \sum_{k=1}^{\infty} \left( \sum_{n=1}^{\infty} \frac{x^{2n}}{k^{2n}} \right) = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} \frac{x^{2n}}{k^{2n}} \right) = \sum_{n=1}^{\infty} \frac{\zeta(2n)x^{2n}}{n(n+1)}.$$

Using the lemma, it follows that,

$$\frac{1}{x} - \pi \cot(\pi x) = \sum_{k=1}^{\infty} \frac{2x}{k^2 - x^2} = \sum_{n=1}^{\infty} 2\zeta(2n)x^{2n-1}.$$

Since the functions $x \mapsto 2\zeta(2n)(1 - x^2)x^{2n-1}$ are positive on the interval $(0, 1)$, we can write :

$$\int_{0}^{1} \left( \sum_{n=1}^{\infty} 2\zeta(2n)(1 - x^2)x^{2n-1} \right) dx = \sum_{n=1}^{\infty} 2\zeta(2n) \int_{0}^{1} (1 - x^2)x^{2n-1} dx = \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(n+1)}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(n+1)} = \int_{0}^{1} (1 - x^2) \left( \frac{1}{x} - \pi \cot(\pi x) \right) dx = I.$$

To evaluate the integral $I$ we start by an integration by parts :

$$\int_{0}^{1} (1 - x^2) \left( \frac{1}{x} - \pi \cot(\pi x) \right) dx = \left[-(1 - x^2) \ln \left( \frac{\sin(\pi x)}{\pi} \right) \right]_{0}^{1} + \int_{0}^{1} \frac{2x(\ln x - \ln \sin(\pi x))dx}{x} = \ln \pi + \int_{0}^{1} \frac{2x \ln x dx}{x} - \int_{0}^{1} \frac{2x \ln \sin(\pi x) dx}{x}.$$
Clearly,  

\[ J = \int_0^1 2x \ln x \, dx = [x^2 \ln x]_0^1 - \int_0^1 x \, dx = -\frac{1}{2}. \]

On the other hand, since by the change of variables \( t = 1 - x \) we have

\[ \int_0^1 x \ln \sin(\pi x) \, dx = \int_0^1 (1 - t) \ln(\pi t) \, dt \]

we conclude that,

\[ K = \int_0^1 2x \ln \sin(\pi x) \, dx = \int_0^1 x \ln \sin(\pi x) \, dx + \int_0^1 (1 - x) \ln \sin(\pi x) \, dx = \int_0^1 \ln \sin(\pi x) \, dx. \]

Now,

\[
\begin{align*}
K &= \int_0^{1/2} \ln \sin(\pi x) \, dx + \int_{1/2}^1 \ln \sin(\pi x) \, dx \\
&= \int_0^{1/2} \ln \sin(\pi x) \, dx + \int_0^{1/2} \ln \sin(\pi (x + 1/2)) \, dx \\
&= \int_0^{1/2} \ln(\sin(\pi x) \cos(\pi x)) \, dx \\
&= \int_0^{1/2} (\ln(2\pi x) - \ln 2) \, dx = -\frac{1}{2} \ln 2 + \frac{1}{2} \int_0^1 \ln \sin(\pi x) \, dx = -\frac{1}{2} \ln 2 + \frac{1}{2} K.
\end{align*}
\]

Hence, \( K = -\ln 2 \), and finally \( I = \ln \pi + J - K = \ln(2\pi) - \frac{1}{2} \), which is the desired result. \( \square \)
Now, let us consider, for

Clearly, it is enough to prove the lemma for positive reals $x$.

**Lemma.** For every nonnegative integer $n$ we have

$$f_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} f_k(x)f_{n-k}(y).$$

**Proof.** We start by giving another formula for $f_n$. Using the well-known Gamma function, we note that for $i \geq 0$ and $x > 0$, we have $\prod_{j=0}^{i-1} (x+j) = \Gamma(x+i)/\Gamma(x)$, so that

$$f_n(x) = \frac{1}{\Gamma(x)} \sum_{i=0}^{n} \binom{n}{i} (-x)^{n-i} \int_0^\infty t^{i+x-1}e^{-t}dt = \frac{1}{\Gamma(x)} \int_0^\infty \left( \sum_{i=0}^{n} \binom{n}{i} (-x)^{n-i}t^i \right) t^{x-1}e^{-t}dt$$

$$= \frac{1}{\Gamma(x)} \int_0^\infty (t-x)^n t^{x-1}e^{-t}dt.$$

Clearly, it is enough to prove the lemma for positive reals $x$ and $y$. Now for $n \geq 0$, $x > 0$ and $y > 0$, we have

$$\sum_{k=0}^{n} \binom{n}{k} f_k(x)f_{n-k}(y) = \frac{1}{\Gamma(x)\Gamma(y)} \int_0^\infty \int_0^\infty \left( \sum_{k=0}^{n} \binom{n}{k} (t-x)^k(s-y)^{n-k} \right) t^{x-1}s^{y-1}e^{-t-s}dt ds$$

$$= \int_0^\infty \int_0^\infty (t+s-x-y)^n t^{x-1}s^{y-1}e^{-t-s}dt ds$$

Now, by the change of variables $t = u(1-v), s = uv$ we can write

$$\sum_{k=0}^{n} \binom{n}{k} f_k(x)f_{n-k}(y) = \frac{1}{\Gamma(x)\Gamma(y)} \int_0^1 \int_0^1 (u-x-y)^n u^{x+y-1}e^{-u(1-v)x-y-1}du dv$$

$$= \frac{1}{\Gamma(x)\Gamma(y)} \int_0^1 (1-v)^{x+y-1}v^{y-1}dv \left( \int_0^\infty (u-x-y)^n u^{x+y-1}e^{-u}du \right)$$

$$= \frac{1}{\Gamma(x+y)} \int_0^\infty (u-x-y)^n u^{x+y-1}e^{-u}du = f_n(x+y)$$

where we used the fact $\int_0^1 (1-v)^{x-1}v^{y-1}dv = \Gamma(x)\Gamma(y)/\Gamma(x+y)$. This completes the proof of the lemma.

Now, let us consider, for $n \geq 1$, the following property:

$\mathbb{P}_n$ : • The polynomial $f_{2n}$ has degree $n$ and the coefficient $\lambda_n$ of $x^n$ in $f_{2n}$ is $\prod_{j=1}^{n}(2j-1)$.
• The polynomial $f_{2n+1}$ has degree $n$ and the coefficient $\mu_n$ of $x^n$ in $f_{2n+1}$ is $\frac{2}{5} n \prod_{j=1}^{n+1}(2j-1)$.
We will prove by induction that $P_n$ is true for every $n \geq 1$.

$\square$ $P_1$ is true, since, $f_2(x) = x$ and $f_3(x) = 2x$.

$\square$ Consider $n > 1$, and assume that $P_k$ is true for $1 \leq k < n$. By the lemma, and using the fact that $f_0 = 1$ and $f_1 = 0$, we have

$$f_{2n}(x + 1) - f_{2n}(x) = \sum_{k=0}^{2n-2} \binom{2n}{k} f_{2n-k}(1)f_k(x).$$

Now, using the induction hypothesis, we conclude that

$$\deg(f_{2n}(x + 1) - f_{2n}(x)) \leq \max\{\deg(f_k) : 0 \leq k \leq 2n - 2\} = n - 1,$$

and that the coefficient of $x^{n-1}$ in $f_{2n}(x + 1) - f_{2n}(x)$ is the same as the coefficient of $x^{n-1}$ in the polynomial $\left(\binom{2n}{2}\right) f_2(1)f_{2n-2}(x)$, i.e. $n(2n - 1)\lambda_{n-1}$. This means that the leading monomial in $f_{2n}(x + 1) - f_{2n}(x)$ is $n(2n - 1)\lambda_{n-1}x^{n-1}$, and consequently, that the leading monomial in $f_{2n}(x)$ is $(2n - 1)\lambda_{n-1}x^n = \lambda_n x^n$.

Similarly, we have

$$f_{2n+1}(x + 1) - f_{2n+1}(x) = \sum_{k=0}^{2n-1} \binom{2n+1}{k} f_{2n+1-k}(1)f_k(x).$$

So, by the induction hypothesis, we obtain

$$\deg(f_{2n+1}(x + 1) - f_{2n+1}(x)) \leq \max\{\deg(f_k) : 0 \leq k \leq 2n - 1\} = n - 1,$$

and the coefficient of $x^{n-1}$ in $f_{2n+1}(x + 1) - f_{2n+1}(x)$ is the same as the coefficient of $x^{n-1}$ in the polynomial $\left(\binom{2n+1}{2}\right) f_2(1)f_{2n-1}(x) + \left(\binom{2n+1}{3}\right) f_3(1)f_{2n-2}(x)$, i.e.

$$n(2n + 1)\mu_{n-1} + 2\binom{2n - 1}{3}(2n+1)\lambda_{n-1} = \frac{2}{3}n^2 \prod_{j=1}^{n+1} (2j - 1).$$

This means that the leading monomial in $f_{2n+1}(x + 1) - f_{2n+1}(x)$ is $\frac{2}{3}n^2 \prod_{j=1}^{n+1} (2j - 1) x^{n-1}$, and consequently, that the leading monomial in $f_{2n+1}(x)$ is $\frac{2}{3}n \left(\prod_{j=1}^{n+1} (2j - 1)\right) x^n = \mu_n x^n$. This achieves the proof of $P_n$, and concludes the proof by induction. $\square$